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The Effect of Replication on Queueing Simulations

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THE EFFECT OF REPLICATION ON QUEUEING SIMULATIONS

BY

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THE EFFECT OF REPLICATION ON QUEUEING SIMULATIONS

This thesis is approved as a creditable and independent investigation by a candidate for the degree, Master of Science, and is acceptable as meeting the thesis requirements for this degree, but without implying that the conclusions reached by the candidate are necessarily the conclusions of the major department.

Thesis Adviser Date

Head, Mechanical Engineering Department Date

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NOMENCLATURE

λ - Mean arrival rate

μ - Mean service rate

ρ - Utilization factor

M - Number of parallel channels

N -Sample size, i.e. number of units

n - Number of replications

W_q - Mean Waiting time

CW_q - Cumulative average of mean waiting time

L_q - Mean queue length

CL_q - Cumulative average of mean queue length

W - Mean time spent in the system

CW - Cumulative average of the mean time spent in the system

L - Mean number of units in the system

CL - Cumulative average of the mean number of units in the system

CHAPTER I

INTRODUCTION

Many industrial systems may be characterized as an arrival of some type of unit (such as sales orders, batches of raw material to be transformed into parts or broken machines) to a system of servicing stations*. In the service stations, the units are serviced in some manner. Thus, the sales order requires checking and preparation of additional paper work; the materials to be transformed require the machines upon which the production process takes place, and the broken machine requires the repair personnel. Waiting lines or queues may form due to lack of control over either the rate of arrival of the units or the amount of service time required per unit or both. In many cases, the time interval between the arrival of units and the time to service the arrivals are not constant, but involve distributions of times from which values randomly occur.

The solution to a queueing problem is essentially economic, leading to a balance between the cost of waiting for the unit, and the cost of idle time for the service stations in the system. The costs associated with waiting may include the loss of customers (if they must wait) as well as in-process inventory costs such as

*A partial list of the terms used in Queueing Theory and Simulation is given in Appendix A.

storage, handling, depreciation, and deterioration. The idle time cost includes the fixed cost of the facility for the non-productive period. However, before the economic evaluation can be made, the system must be described in terms of the variables of the queueing systems, and the possible decisions articulated.

In general, six variables must be ascertained. These are:

1. Population size
2. System configuration, i.e. series and/or parallel channels of service stations
3. Maximum queue length before each station
4. Queue discipline
5. Interarrival time distribution
6. Service time distribution

Some of the possible classificiations for each of these variables are shown in Figure 1.1 [1].* Rigorous statistical equations have been developed for specific cases for many of the combination of variables in Figure 1.1. However, many realistic problems are too complex to be solved in a satisfactory manner by the application of queueing theory and the resulting statistical equations. Because of the difficulties inherent in the mathematical derivations of the more complex queueing situations, another approach must be found. One solution is to simulate the system.

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*The number in the bracket is the reference number.

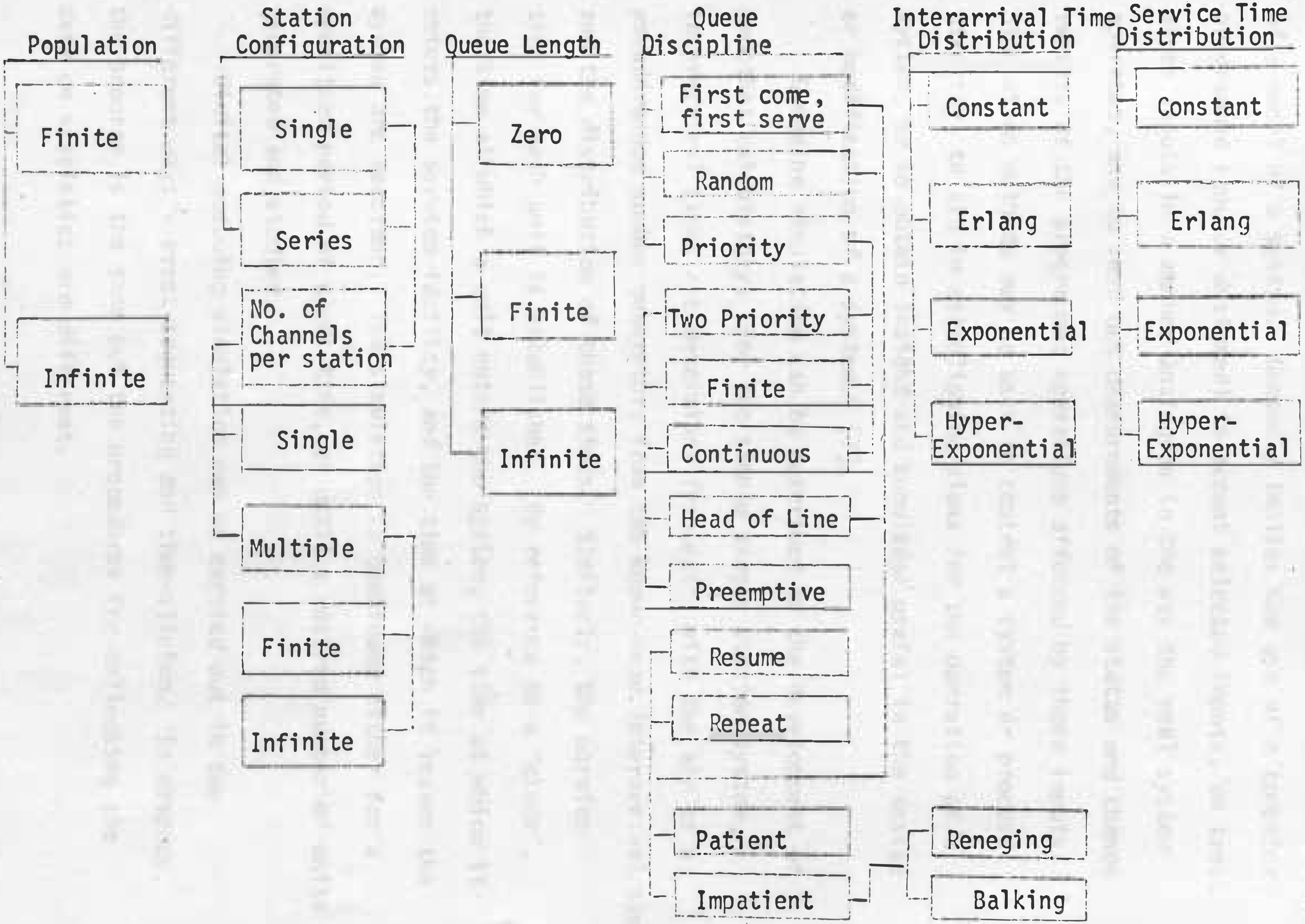

Figure 1.1. Variables in Queueing Situations

One definition of simulation is: "The design and operation of a model of a system. Commonly implies the use of a computer, programmed (and/or designed) to accept selected inputs, to treat those inputs in a manner analogous to the way the real system operates, and to read out measurements of the status and change results of the programmed operations affected by those inputs. Simulation methods may be used to control a system or process directly, to aid in establishing plans for the operation of a system, or to obtain insight and knowledge useful in the design or modification of a system." [7].

Queueing simulation can be described as the development of hypothetical history. For the simple single station system, interarrival times are determined frequently with the aid of a pseudorandom number generator, from the known mean interarrival time and the distribution of those times. Similarly, the service time for each unit is established. By reference to a "clock", the time at which a unit enters the system, the time at which it enters the service facility, and the time at which it leaves the system are recorded. The simulation is continued either for a specified period of clock time, or until a desired number of units are generated and studied.

Digital queueing simulation can be carried out in two different ways - event-sequencing and time-slicing. In essence, the program is the same but the procedures for collecting the data on statistics are different.

In the event-sequencing method, the time of occurrence of the various events are recorded where an event is defined as a change in the state of the system. The simulation program proceeds from one event to the next, maintaining a record of the system state at all times along with a record of the elapsed time between events. This method basically keeps record of each and every event occurring during the entire period of simulation. Though it is an advantage of having the whole history of simulation recorded, it will be cumbersome and inconvenient when a large number of units are likely to be in the system at one time.

In the time-slicing method, the program again precedes as for the event sequencing, but some or all of the statistics are maintained by sampling the system at specific times rather than the keeping of continuous records. In this type of simulation, some information is likely to be lost. If, for example, an event occurs between two successive sampling points, the exact time of occurrence is not known. The program assumes that it occurs at either the previous or the following sample point. For fixed intervals, an average bias of one half of the interval is introduced. If the sampling interval is made smaller, the error introduced becomes smaller. The event-sequencing program may also be considered as the time slicing program with infinitely small sampling interval. These two methods are best shown by an example of a simple case:

Repair orders arrive with a Poisson rate of arrival distribution and a mean of 3 per hour. The length of time taken to perform the repair function has an exponential distribution, with a mean time of 15 minutes.

Therefore,

λ , average arrival rate = 3/hour

μ , average service rate = 4/hour

since $\frac{1}{\mu}$ = 15 minutes

let

RT = randomly selected interarrival time

ST = randomly selected service time

RN = Random Number

Then,

$$RN = e^{-\lambda (RT)}$$

$$RT = \frac{1}{\lambda} \ln \frac{1}{RN}$$

Similarly, $$ST = \frac{1}{\mu} \ln \frac{1}{RN}$$

For each unit, a random number is generated to get the inter-arrival time and another to get the service time. Twenty-five units are simulated through a single service facility. Table 1.1 shows the data collected by using the event-sequencing method and Table 1.2 shows the data obtained by using the time-slicing methods. The statistics that are calculated from these data are:

1. Average waiting time for a unit - W_q
2. Average time a unit spends in the system - W
3. Average queue length - L_q
4. Average numbers of units in the system - L

Table 1.1

Simulation results of 25 units by event-sequencing method

Cumulative Arrivals	Random Number	Interarrival Time	Arrival Time	Waiting Time	Random Number	Service Time	Time of Entering Service	Time of Completion of Service	Total Time In System
1	0.09	0.80	0.080	0.00	0.98	0.01	0.80	0.81	0.01
2	0.54	0.21	1.01	0.00	0.70	0.09	1.01	1.10	0.09
3	0.43	0.28	1.29	0.00	0.39	0.23	1.29	1.42	0.23
4	0.73	0.11	1.39	0.13	0.05	0.76	1.52	2.28	0.89
5	0.49	0.24	1.63	0.65	0.77	0.07	2.28	2.35	0.72
6	0.34	0.36	1.99	0.36	0.16	0.45	2.35	2.80	0.81
7	0.67	0.14	2.13	0.67	0.10	0.57	2.80	3.37	1.24
8	0.95	0.02	2.15	1.23	0.14	0.50	3.37	3.87	1.73
9	0.73	0.11	2.25	1.62	0.89	0.03	3.87	3.90	1.65
10	0.78	0.08	2.34	1.56	0.13	0.52	3.90	4.42	2.08
11	0.13	0.68	3.02	1.40	0.74	0.08	4.42	4.49	1.47
12	0.78	0.08	3.10	1.39	0.28	0.32	4.49	4.82	1.71
13	0.52	0.22	3.32	1.49	0.04	0.78	4.82	5.60	2.27
14	0.10	0.78	4.10	1.49	0.77	0.07	5.60	5.66	1.56
15	0.91	0.03	4.14	1.53	0.22	0.38	5.66	6.05	1.91
16	0.60	0.17	4.30	1.74	0.37	0.25	6.05	6.29	1.99
17	0.42	0.29	4.59	1.70	0.30	0.30	6.29	6.59	2.01
18	0.11	0.75	5.34	1.26	0.45	0.20	6.59	6.79	1.46
19	0.82	0.07	5.40	1.39	0.98	0.01	6.79	6.80	1.40
20	0.98	0.08	5.41	1.39	0.86	0.04	6.80	6.84	1.43
21	0.47	0.25	5.66	1.18	0.35	0.26	6.84	7.10	1.44
22	0.01	1.57	7.23	0.00	0.35	0.26	7.23	7.49	0.26
23	0.84	0.06	7.29	0.20	0.99	0.01	7.49	7.50	0.21
24	0.97	0.01	7.30	0.20	0.74	0.08	7.50	7.58	0.27
25	0.22	0.50	7.80	0.00	0.57	0.14	7.80	7.94	0.14

Table 1.2

Simulation results of 25 units by time-slicing method

Clock Time	Time of Next Arrival	Cumulative Arrivals	Number In System	Number In Queue	Waiting Time	Total Time
0.0	0.23	1	1	0	0.0	0.378
0.50	0.52	3	2	1	0.0	0.378
1.00	1.05	9	5	4	0.215	0.471
1.50	2.25	12	7	6	0.469	0.635
2.00	2.25	12	6	5	0.450	0.807
2.50	2.96	15	5	4	1.222	1.516
3.00	3.57	16	5	4	1.475	2.135
3.50	3.57	16	4	3	1.006	1.381
4.00	4.10	17	3	2	1.228	1.386
4.50	5.16	21	5	4	0.469	1.436
5.00	5.16	21	4	3	0.900	1.211
5.50	5.55	23	5	4	0.900	1.211
6.00	6.27	25	7	6	1.077	1.769

The statistics are calculated as follows:

$$W_q = \frac{\sum \text{(waiting time)}}{\text{Number of services}}$$

$$W = \frac{\sum \text{(Total time in system)}}{\text{Number of services}}$$

$$L = \sum n P(n)$$

where $P(n)$ is the state probability and

= 0,1,2, ...

$$L_q = L - \rho$$

For this particular case, satisfactory mathematical equations are readily available for the desired statistics [9].

Average waiting time for a unit,

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)} \qquad (1.1)^*$$

$$= \frac{3}{4(4-3)}$$

$$= 0.75 \text{ hours}$$

Average queue length,

$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} \qquad (1.2)$$

$$= \frac{9}{4(4-3)}$$

$$= 2.25$$

Average time a unit spends in the system,

$$W = \frac{1}{\mu-\lambda} \qquad (1.3)$$

$$= \frac{1}{(4-3)}$$

$$= 1 \text{ hour}$$

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*The equations 1.1 through 1.4 are from reference 9.

Average number of units in the system,

$$L = \frac{\lambda}{\mu - \lambda} \tag{1.4}$$

$$= \frac{3}{4-3}$$

$$= 3$$

A comparison of the results obtained from simulations and that from mathematical equations is given in Table 1.3. Simulation results differ from mathematical ones by a large percentage. The reason for such a difference is that the data in Table 1.1 and in Table 1.2 are small samples of the simulated universe. Since these samples are very small, it is not surprising that it gives results which differ by a considerable amount from the true values obtained by using the mathematical equations.

The convergence of the sample average towards the true value is called stochastic convergence and is very slow in nature. A measure of the amount of random fluctuation inherent in any sample value is its standard deviation. If s is the standard deviation of a single observation, and $s_{\bar{x}}$ is the standard deviation of the average of n observations, then $s_{\bar{x}} = \frac{s}{\sqrt{n}}$.

Thus the standard deviation of the average is inversely proportional to the square root of the sample size. To reduce the standard deviation to one half of its initial value, the sample size must be four times as large as the orginal sample. Because of the slowness of the stochastic convergence, increase in sample size is not likely to be the optimum approach.

Table 1.3

Comparison of simulation results with mathematical results

$M=1 \qquad \rho=0.75$

Method	W_q	L_q	W	L
Mathematical	0.750	2.250	1.000	3.000
Event-Sequencing	0.903	3.320	1.159	4.028
Error	20.4%	47.5%	15.9%	34.2%
Time-Slicing	0.821	3.538	1.151	4.538
Error	9.6%	57.2%	15.1%	51.3%

The sample size required to give accurate estimates of the queueing statistics may be different for different queueing situations. Some of the factors affecting the estimates may be:

1. Interarrival time distribution and service time distribution
2. Number of channels
3. Utilization factor
4. Truncation of the function for zero value.

For a given model the first two factors are fixed. Since the simulation starts from initially idle conditions (zero units in the system), there will be a transient period before the system reaches its long term mean values. All of this sample (data in Tables 1.1 or 1.2) may represent the transient period.

Morse [8], in the chapter, "An Example of Transient Behaviour" has discussed the transient period for one channel exponential system. He defines the term "relaxation time" as the time for the transient to die down to (1/e) of its initial value. For the case where a long queue is allowed, he has shown that the relaxation time is approximately $\frac{1}{(\sqrt{\mu} - \sqrt{\lambda})^2}$.

If λ =3/hour, and μ =4/hour, the relaxation time will be equal to 13.9 hours.

In terms of the arriving units, the transient reaches 1/e of initial value after 41.7 units. If a value of (2/e) will reduce the transient sufficiently, then the entire sample in Table 1.1 and Table 1.2 is from the transient period. Further, it may be seen that the transient period will be long when the utilization factor is near one $(\lambda \rightarrow \mu)$. Hence, larger samples are required for higher utilization factors, if the transient period must be included.

When the interarrival time distribution and the service time distribution are other than the constant time distribution, the interarrival time between two units and the service time of a unit may have a zero value too. In practice, it is impossible to have a zero service time though it is possible to have zero interarrival time in cases such as bulk arrivals. Therefore, by using a zero-truncated approximation to a distribution, the results will be in error. The pseudorandom number subroutine used by Gould [4] gives uniformly distributed random numbers between .0000001 and .9999999. Therefore, the distribution is truncated. The generator truncation is only 1 part in 10 million and hence the error is negligible.

Further, in the Table 1.3, it may also be seen that the percentage of error is different for the various statistics. This may mean that different sample sizes will be required for different statistics to obtain the same degree of accuracy. Very large samples, even on high speed digital computers, tend to be expensive to obtain. Hence, it is desirable to determine the minimum sample size which will give reasonable results for the statistics involved.

CHAPTER II

LITERATURE REVIEW

With the growing use of simulation, one would expect a body of literature about the minimum sample size for statistics. In fact, there are relatively few publications in the area of estimating the sample size required to obtain simulation results within a desired degree of accuracy and for an established confidence level. Of these, only a few deal with methods applicable to queueing problems. No general solution which will give satisfactory results for all of the statistics was found. The methods and equations found in the literature are applicable only for a single statistic and for a limited queueing system.

Hammersely and Handscomb [5] discuss a number of Monte-Carlo techniques. Some of them are stratified sampling, control variates, and antithetic variates. They have stated that a greater reduction in the variance of the estimate will be achieved for a given sample size if any of these Monte Carlo techniques is used. However, the application of these techniques to queueing systems is not discussed and has not been found in literature. All of these techniques are described in connection with their application to nuclear physics.

Ehrenfeld and Ben-Tuvia [2] enumerate a number of methods for improving the efficiency of statistical simulation of complex systems. Efficiency is defined as the ratio of the percentage reduction in the volume of sampling required to attain a fixed confidence as compared to the volume of completely random sampling.

Thus, if N_R is the volume of sampling required for the standard method of random sampling, the N_M is the volume of sampling required for some alternative method of sampling, then the efficiency of the alternative method is defined as $100(N_R-N_M)/N_R$ percent.

Ehrenfeld and Ben-Tuvia discuss the use of sampling techniques such as: proportional sampling, fixed sequence sampling, importance sampling, and the use of concomitant information to obtain accurate results. The efficiency of the simulation can be increased through the adaptation of experimental design principles if any qualitative knowledge surrounding the problem area is known. For example, in the case of estimating the mean waiting time, it is known that the mean waiting time will be large if the service times are large and the interarrival times are small. In the end, they have indicated the use of the previously mentioned sampling techniques for estimating the mean waiting time alone.

Healy [6] has arrived at a set of relationships to calculate the sample size which would give an accurate and reliable estimate of L, the mean number of units in the system. He defines accuracy and reliability as: Accuracy is the closeness with which the estimated value approaches the true value; and Reliability is the degree of confidence which may be placed in the estimated value.

Healy used a single channel service station with an infinite queue, and first come, first serve queue discipline as his model.

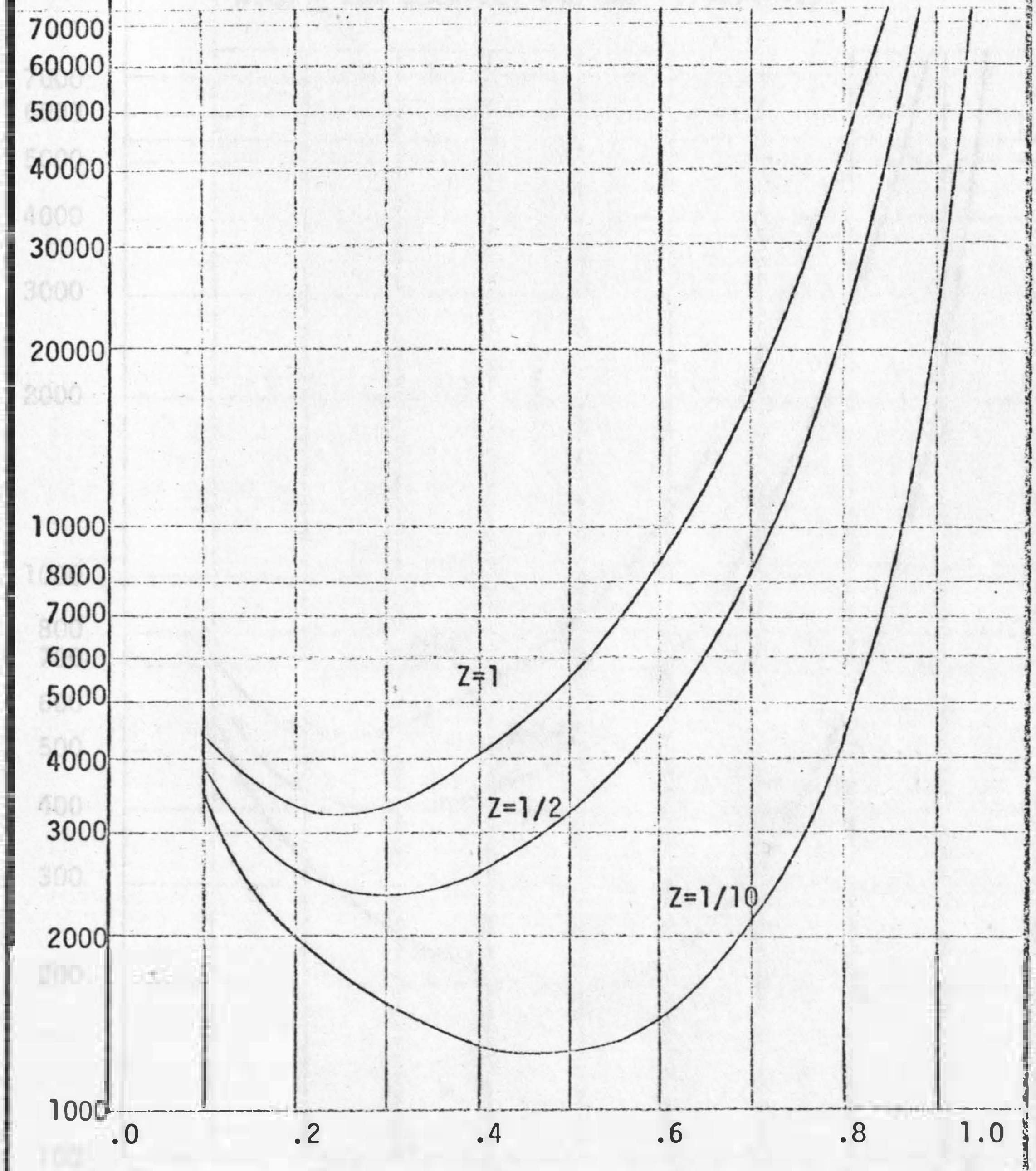
The rate of arrival is Poisson and the service time distribution is exponential. He has drawn two graphs developed from the sets of relationships he derived. These graphs, shown in Figures 2.1, and 2.2, show the minimum sample size (N) as a function of the utilization factor ρ and a factor (Z) related to the rate at which sample observations are made. Sample observations are taken at a rate of λz where λ is the arrival rate. For Z=1, the mean number in the system is counted immediately after every arrival. For Z=1/2 the number in the system is counted only after every other arrival. One may note that he is using the random time period time-slicing technique. In Figure 2.1, for ρ =0.75 and Z=1, a sample of approximately 20,000 is required. That is, at least 20,000 arriving units must be simulated. On the other hand, for ρ =0.75 and sample observations taken immediately after every other arrival (i.e. Z=1/2), the system will be sampled approximately 12,000 times or that at least 24,000 arriving units must be simulated. The size of sample required is highly dependent upon the degree of accuracy and the specified reliability. If one or both of these requirements is reduced, the sample size is also reduced.

Gafarian and Ancker, Jr., [3] have discussed the sampling interval used in the time-slicing program. They define efficiency as the ratio $\hat{\theta}_2 / \hat{\theta}_1$ where

$\hat{\theta}_2$ = variance of the estimate of the mean in the case of event-sequencing method and

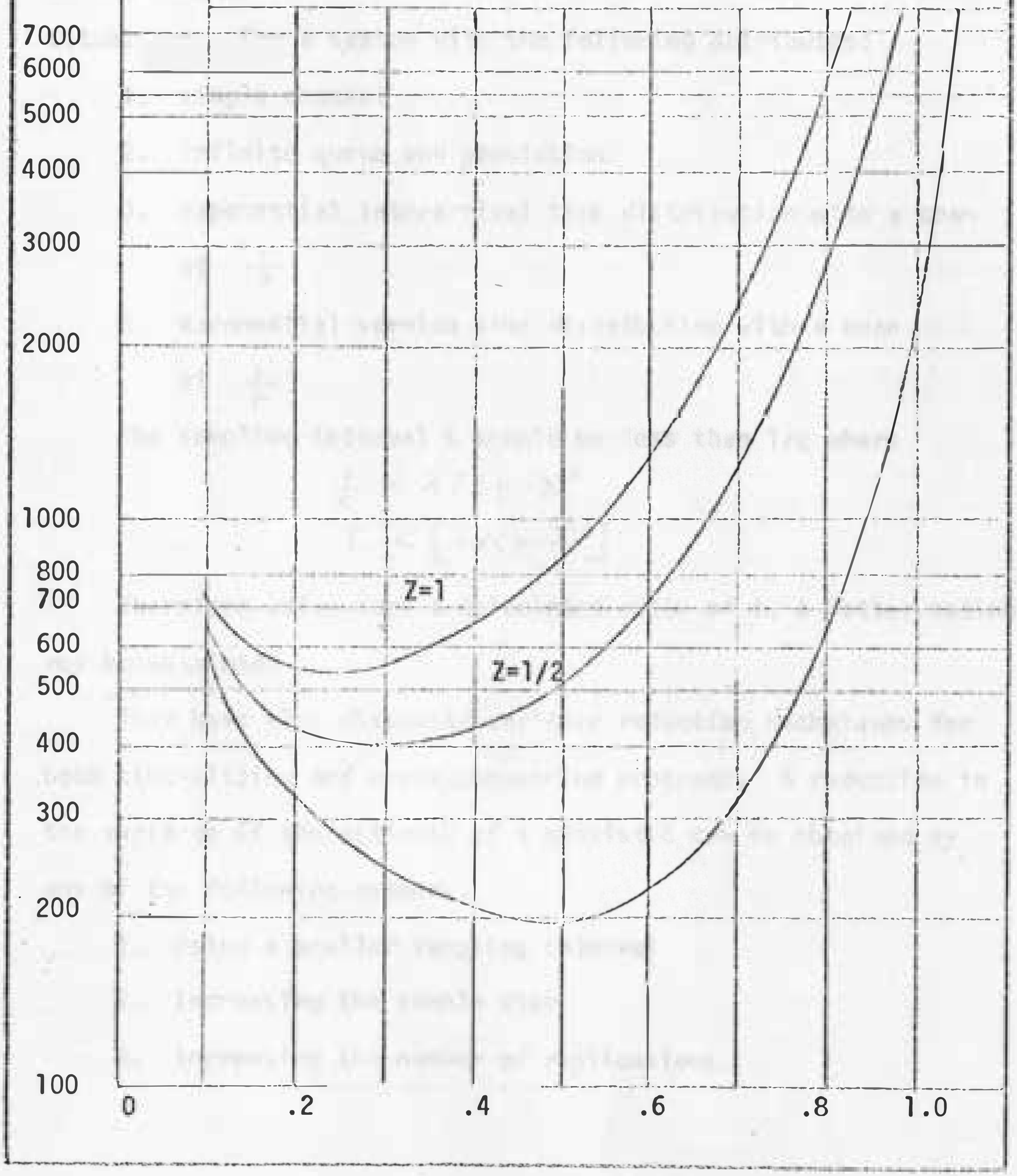
$\hat{\theta}_1$ = variance of the estimate of the mean in the case of time slicing method.

Figure 2.1. Sample size, N, as a function of ρ and Z for estimating the mean number in the system with $\pm$ 10% accuracy and 95% reliability.*



*Reproduced from Reference 6.

Figure 2.2. Sample size, N, as a function of ρ and Z for estimating Mean number in the system with $\pm$ 25% accuracy and 95% reliability.*



*Reproduced from Reference 6.

The time-slicing is efficient when this ratio is near one. They have shown that for any reasonable efficiency (say > 0.90) and for practically any sample size, the sampling interval t must be less than the constant 1/c, which is different for various situations. For a system with the following attributes:

1. single channel
2. infinite queue and population
3. exponential interarrival time distribution with a mean of $\frac{1}{\lambda}$
4. exponential service time distribution with a mean of $\frac{1}{\mu}$

The sampling interval t should be less than 1/c where

$$\frac{1}{c} = \lambda/(\mu-\lambda)^2$$

$$t < [\lambda/(\mu-\lambda)^2]$$

Therefore using such a calculated value of t, a better estimate may be obtained.

They have also discussed variance reduction techniques for both time-slicing and event-sequencing programs. A reduction in the variance of the estimate of a statistic can be obtained by any of the following methods.

1. Using a smaller sampling interval
2. Increasing the sample size
3. Increasing the number of replications.

As it was described earlier, a reduction in the sampling interval beyond a certain point ($\frac{1}{c}$) will not increase the efficiency (reduce the variance) to a great extent. It is shown that replication gives the maximum reduction in the variance. However, Gafarian and Ancker do not indicate the optimum number of replications or the sample size of each replication. Hence, it is necessary to determine the optimum number of replications and the sample size for each replication.

This survey of literature indicates that considerable research needs to be done in this area. It would be desirable to estimate the minumum sample size-number of replications required to get simulation results to a given degree of accuracy.

CHAPTER III

MODEL AND PROCEDURE

The purpose of this research is to investigate the sample size-replication combination which will give accurate and reliable estimates of the queueing statistics. A number of queueing situations will be simulated and the statistics will be estimated. These estimates will be compared with the true values for accuracy and thereby the minimum sample size-replication combination will be found for the desired accuracy.

Computer programs for simulating queueing systems can be written in any one of the well known general-purpose compiler languages such as FORTRAN, ALGOL, OR PL/1. In addition, a number of special purpose simlation or macro languages have been developed in the recent years. Some of them are GPSS, SIMSCRIPT, GASP, SIMPAC, DYNAMO and SIMULATE. Although these languages simplify the task of writing simulation programs, they are available only for computers that have greater capacity than the locally available IBM-360/30 computer.

A number of simulation programs written in Fortran IV are available through private files. Of these the one which most closely fits the desired output was written by Gould [4]. His model, written in Fortran IV was therefore used with some minor changes.* The capabilities of this program are shown in Table 3.1

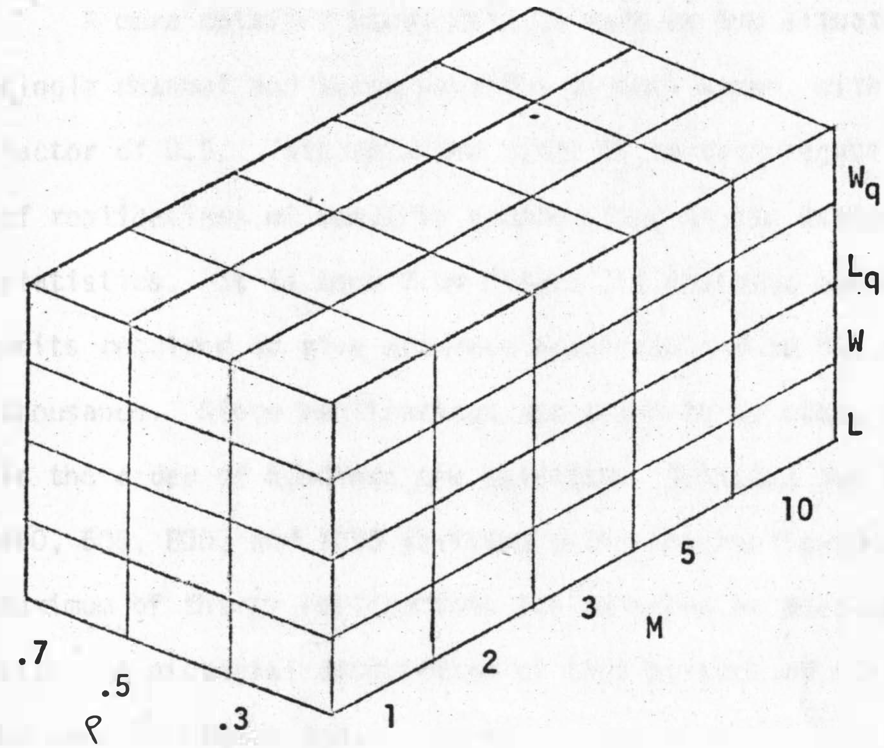
*A listing of the program is given in the Appendix of Reference 1.

Figure 3.1 gives a general description of the experiment. Thirty replications will be made with 1000 units in each. Different seed random numbers will be used for different combinations. For different values of the utilization factor, ρ , different values of the mean service rate, μ will be used by keeping the mean arrival rate λ at a constant value of 5 per time unit. The number of replications (30) is arbitrarily chosen with the hope that it will be possible to obtain estimates accurately with this many replications. The statistics will be estimated at the end of thirty replications. These will be compared with the theoretical values obtained from the relations determined by queueing theory.

Table 3.1

Capability of Gould's program

No.	Factor	Capability
1.	Number of parallel channels	ten
2.	Type of queue	finite as well as infinite
3.	Interarrival time distribution	a. exponential b. constant c. ten celled histogram
4.	Service time distribtuion	a. exponential b. constant c. ten celled histogram
5.	Queue discipline	first come, first serve
6.	Maximum number of units per run	999
7.	Maximum number of replication	thirty



N = 1000

n = 30

Figure 3.1. Possible combinations of the general experiment.

A more detailed study will be made on two situations, namely single channel and three parallel channel cases, with an utilization factor of 0.5. This detailed study is to investigate the effect of replications of specific sample sizes on the estimates of the statistics. It is seen from Figure 2.1 that the number of arriving units required to give accurate estimates are in the order of thousands. Since replications are going to be made, sample sizes in the order of hundreds are selected. Selected are 100, 200, 400, 600, 800, and 1000 arriving units per replication. A maximum of thirty replications are selected as adequate for each sample size. A pictorial description of this portion of the experiment can be seen in Figure 3.2.

The cumulative average of the estimates of the statistics will be plotted. The theoretical values of the statistics will be calculated with the help of the mathematical equations. The criteria are to obtain estimates within $\pm 10\%$ of the theoretical values with 95% confidence on these estimates. The cumulative averages of the estimates at the end of 5, 10, 15, 20, 25 and 30 replications are of particular importance. The procedure for attaining a 95% confidence is shown by an example.

For the case of a single exponential channel with an utilization factor of 0.5, the mean number of units in the system is 1.000. Therefore, for a $\pm 10\%$ accuracy,

upper limit = 1.100

lower limit = 0.900

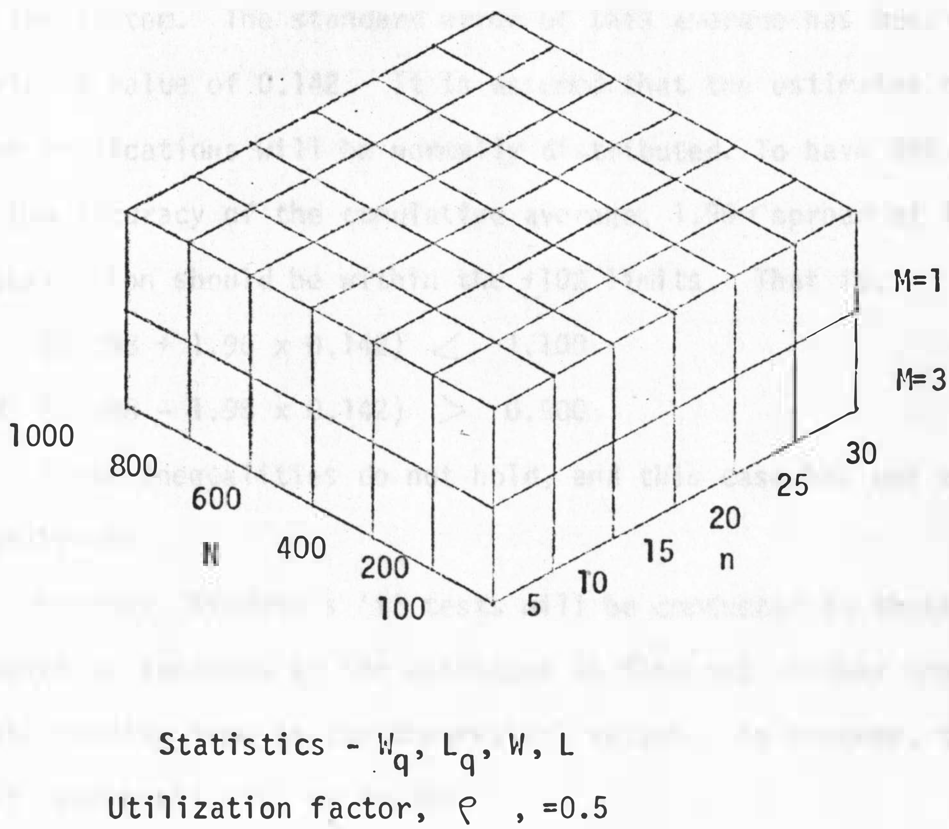


Figure 3.2. Possible combinations of the detailed experiment.

Ten replications of 1000 units each give a value of 0.998 for the cumulative average of the estimate of the mean number of units in the system. The standard error of this average has been calculated giving a value of 0.142. It is assumed that the estimates obtained from replications will be normally distributed. To have 95% confidence on the accuracy of the cumulative average, 1.96σ spread of its distribution should be within the $\pm$10% limits. That is,

$(0.998 + 1.96 \times 0.142) < 1.100$

and $(0.998 - 1.96 \times 0.142) > 0.900$

These inequalities do not hold, and this case has not met the requirement.

Further, Student's 't' tests will be conducted on these cumulative averages of the estimates to find out whether they are statistically same as the theoretical values. In essence, the null hypothesis will be tested.

H_0: the cumulative average = theoretical vlaue

α = level of significance (chosen)

= 0.05

$$t = \frac{\text{cumulative average-theoretical value}}{\text{standard error}}$$

If the calculated value of t is less than the theoretical value, then H_0 is accepted; otherwise rejected.

CHAPTER IV

RESULTS

Simulation runs were made on a number of combinations of utilization factor, sample size, and replication for the system configurations as indicated in Chapter III. The estimates of the statistics were collected for each replication and the averages of these estimates are calculated. Table 4.1 shows that results of the simulation runs for the system configurations shown in Figure 3.1. Theoretical values of the statistics were calculated as shown in Appendix B. These values are included in Table 4.1, and the percentage error in simulation results are also indicated. In most cases, the estimates show a $\pm$ 10% accuracy. Therefore, thirty replications with 1000 units in each are sufficient to obtain estimates with this specified accuracy. Further, it is seen that in many cases the percentage error is zero. Therefore, smaller size may be adequate to achieve $\pm$ 10% accuracy.

The cumulative averages of the estimates of the statistics obtained from larger samples will be more accurate than those from smaller samples. The estimates of the four statistics obtained from each replication and the cumulative averages of the replications for the configurations shown in Figure 3.2 are given in Appendix C from page 56 to page 67. The cumulative averages are shown graphically against replications for each sample size in

Table 4.1A

N=1000 n =30

MEAN WAITING TIME W_q

	M / Method	1	2	3	5	10
0.3	Mathematical	0.026	0.012	0.006	0.002	0.000
	Simulation	0.025	0.012	0.006	0.002	0.000
	Error by Simulation	-3.9%	0.0%	0.0%	0.0%	0.0%
0.5	Mathematical	0.100	0.067	0.047	0.026	0.007
	Simlation	0.100	0.066	0.048	0.027	0.007
	Error by Simulation	0.0%	-1.5%	+2.1%	+3.9%	0.00%
0.7	Mathematical	0.327	0.269	0.230	0.176	0.104
	Simulation	0.357	0.260	0.212	0.175	0.106
	Error by Simulation	+9.2%	-3.4%	-7.8%	-0.6%	+2.0%

Table 4.1B

MEAN QUEUE LENGTH L_q

	M / Method	1	2	3	5	10
0.3	Mathematical	0.129	0.059	0.030	0.009	0.001
	Simulation	0.150	0.064	0.031	0.007	0.001
	Error by Simulation	+16.3%	+8.5%	+3.3%	-22.2%	0.0%
0.5	Mathematical	0.500	0.333	0.237	0.130	0.036
	Simulation	0.541	0.338	0.241	0.132	0.030
	Error by Simulation	+8.2%	+1.5%	+1.7%	+1.5%	-16.7%
0.7	Mathematical	1.633	1.345	1.149	0.882	0.518
	Simulation	1.837	1.297	1.064	0.869	0.522
	Error by Simulation	+12.5%	-3.6%	-7.4%	-1.5%	+0.8%

Table 4.1C

MEAN TOTAL TIME SPENT IN THE SYSTEM W

	Method M =	1	2	3	5	10
0.3	Mathematical	0.086	0.132	0.186	0.302	0.600
	Simulation	0.085	0.134	0.188	0.303	0.600
	Error by Simulation	+1.2%	+1.5%	+1.1%	+0.3%	0.0%
0.5	Mathematical	0.200	0.267	0.347	0.526	1.007
	Simulation	0.201	0.265	0.348	0.528	1.007
	Error by Simulation	+0.5%	-0.8%	+0.3%	+0.4%	0.0%
0.7	Mathematical	0.467	0.547	0.650	0.876	1.504
	Simulation	0.498	0.541	0.629	0.872	1.511
	Error by Simulation	+6.6%	-1.1%	-3.2%	-0.5%	+0.5%

Table 4.1D

MEAN NUMBER IN THE SYSTEM L

	Method M =	1	2	3	5	10
0.3	Mathematical	0.429	0.659	0.930	1.509	3.001
	Simulation	0.505	0.689	0.939	1.504	2.949
	Error by Simulation	+17.7%	+4.6%	+1.0%	-0.3%	-1.7%
0.5	Mathematical	1.000	1.333	1.737	2.630	5.036
	Simulation	1.087	1.354	1.753	2.628	4.977
	Error by Simulation	+8.7%	+1.6%	+0.9%	-0.1%	-1.2%
0.7	Mathematical	2.333	2.745	3.249	4.382	7.518
	Simulation	2.570	2.713	3.171	4.347	7.523
	Error by Simulation	+10.2%	-1.2%	-2.4%	-0.8%	+0.1%

Figures 4.1 through 4.8. The graphs for 400 and 800 sample sized runs are not included since they are basically the same as that of 200 and 1000 units respectively.

The mean value of the statistic fluctuates widely for at least five replications before it begins to approach a final value. The amount and period of fluctuation are different for different statistics and for different sample sizes. In general, the fluctuations in the larger samples tend to damp out faster than those obtained from smaller samples. Indeed, the estimate of the mean queue length for samples of 100, as shown in Figure 4.2, continued to vary for all thirty replications.

The theoretical values of the statistics* were used as the true values. These values for the cases M=1, and M=3 are:

Situation		W_q	L_q	W	L
Single channel,	$\rho = 0.5$	0.100	0.500	0.200	1.000
3 parallel channels,	$\rho = 0.5$	0.047	0.237	0.347	1.737

In Figures 4.1 through 4.8, the theoretical values are shown as straight lines at these values, with the upper limit at 110% of the actual value; and the lower limit at 90% of the actual value. In many cases, the $\pm$ 10% accuracy is achieved within a few replications although the mean of the statistic continued to fluctuate within these limits. Their continued fluctuations may not be critical if the amount of fluctuation is low.

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*Computations to obtain the theoretical values are given in Appendix B.

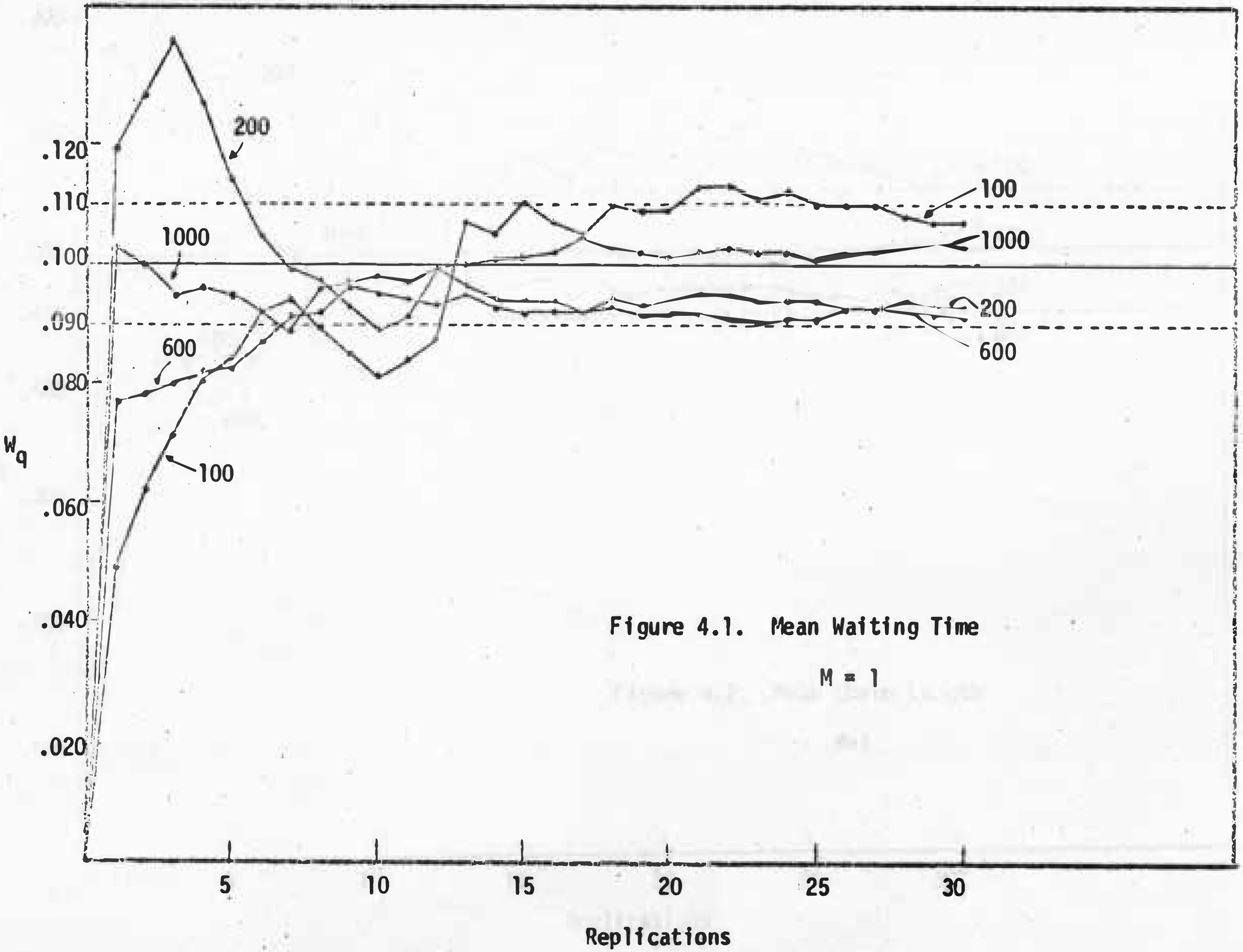

Figure 4.1. Mean Waiting Time

M = 1

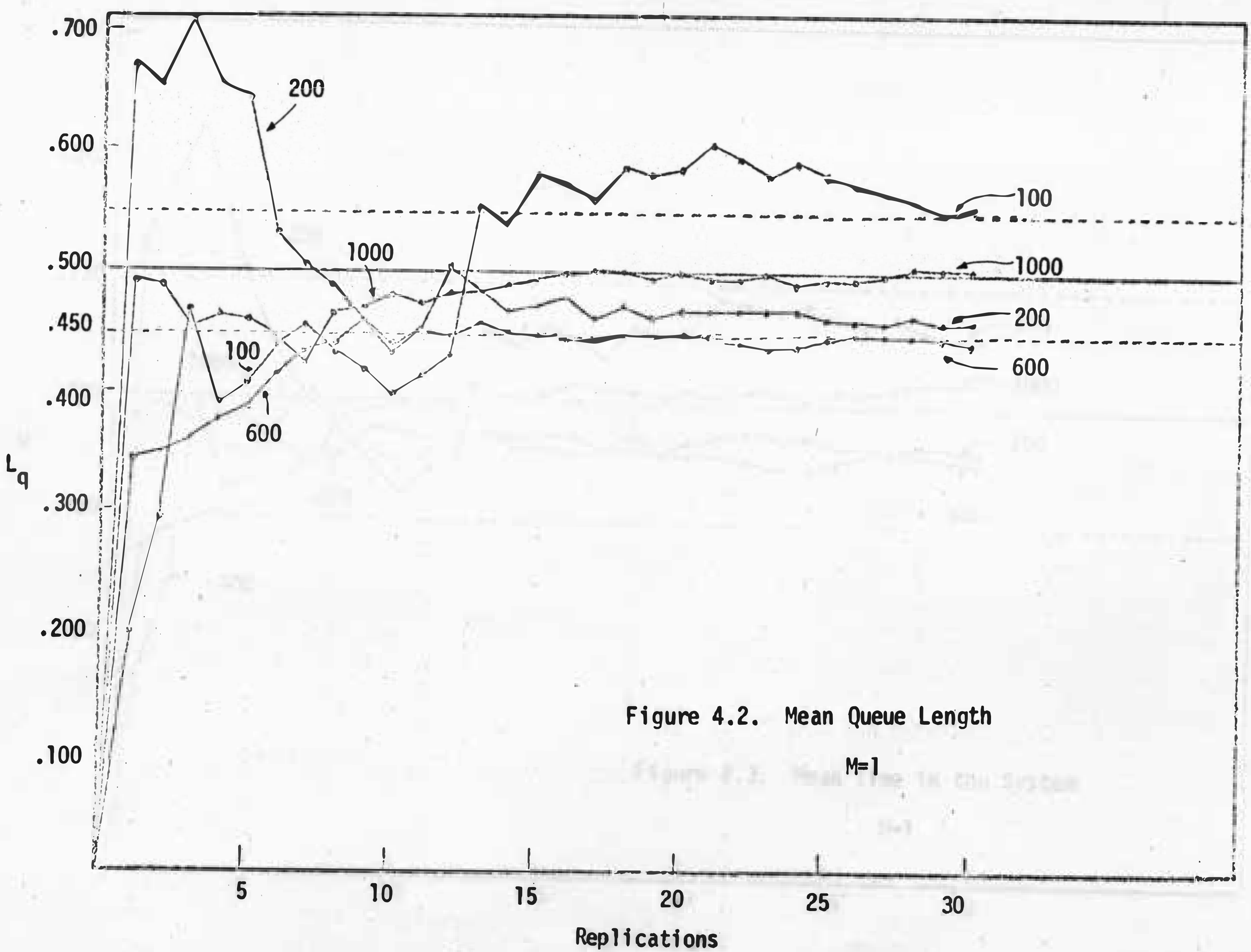

Figure 4.2. Mean Queue Length

M=1

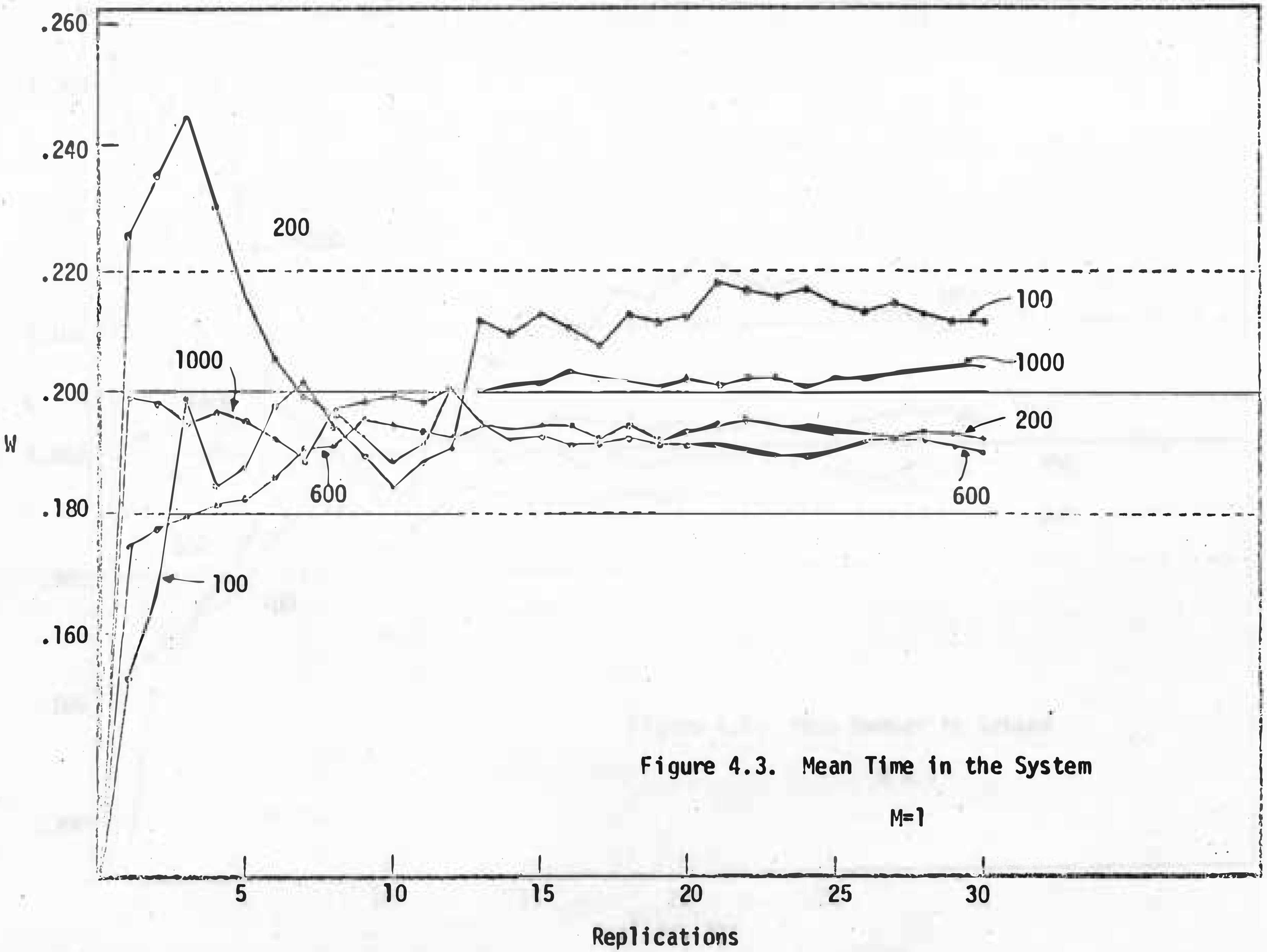

Figure 4.3. Mean Time in the System

M=1

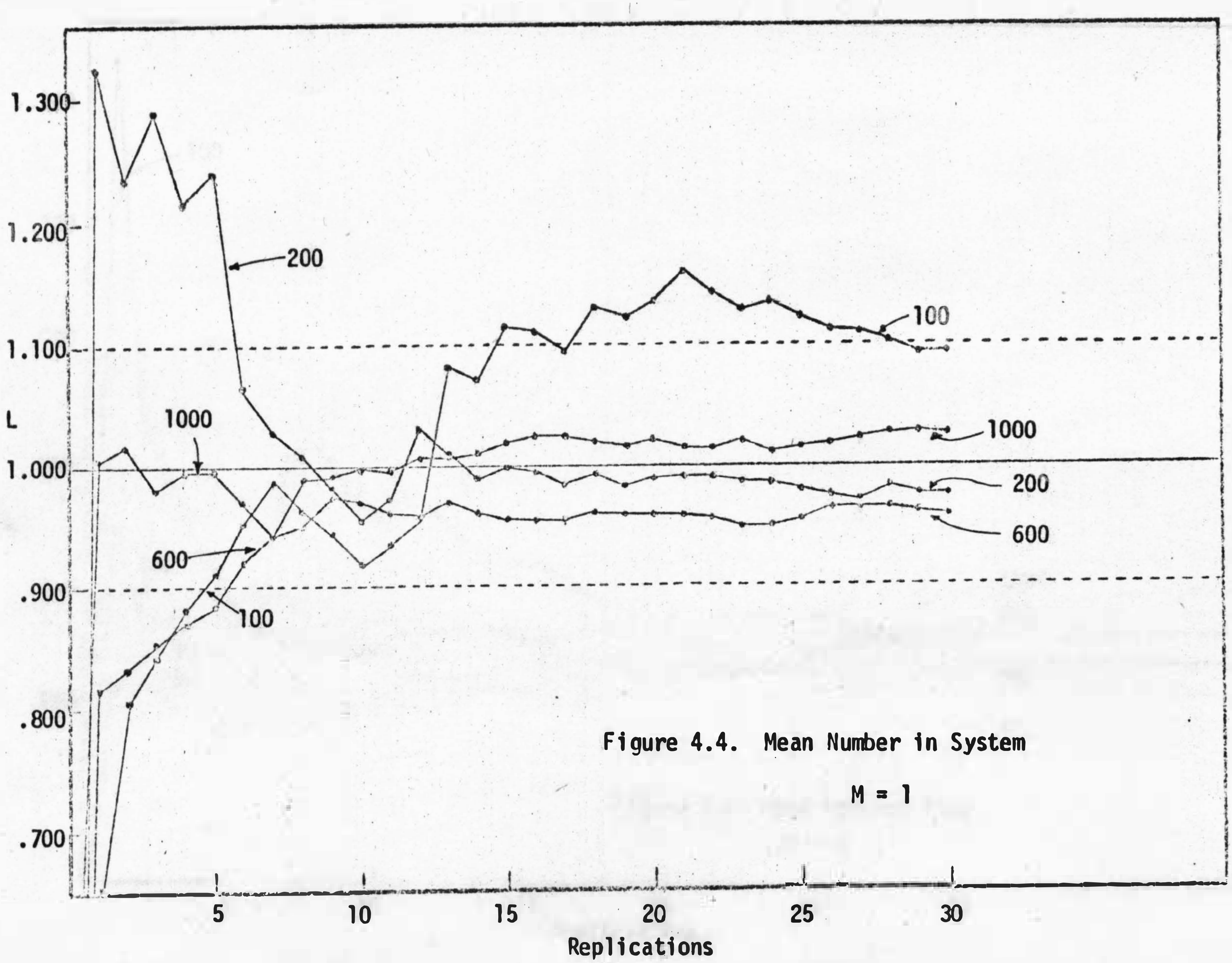

Figure 4.4. Mean Number in System

M = 1

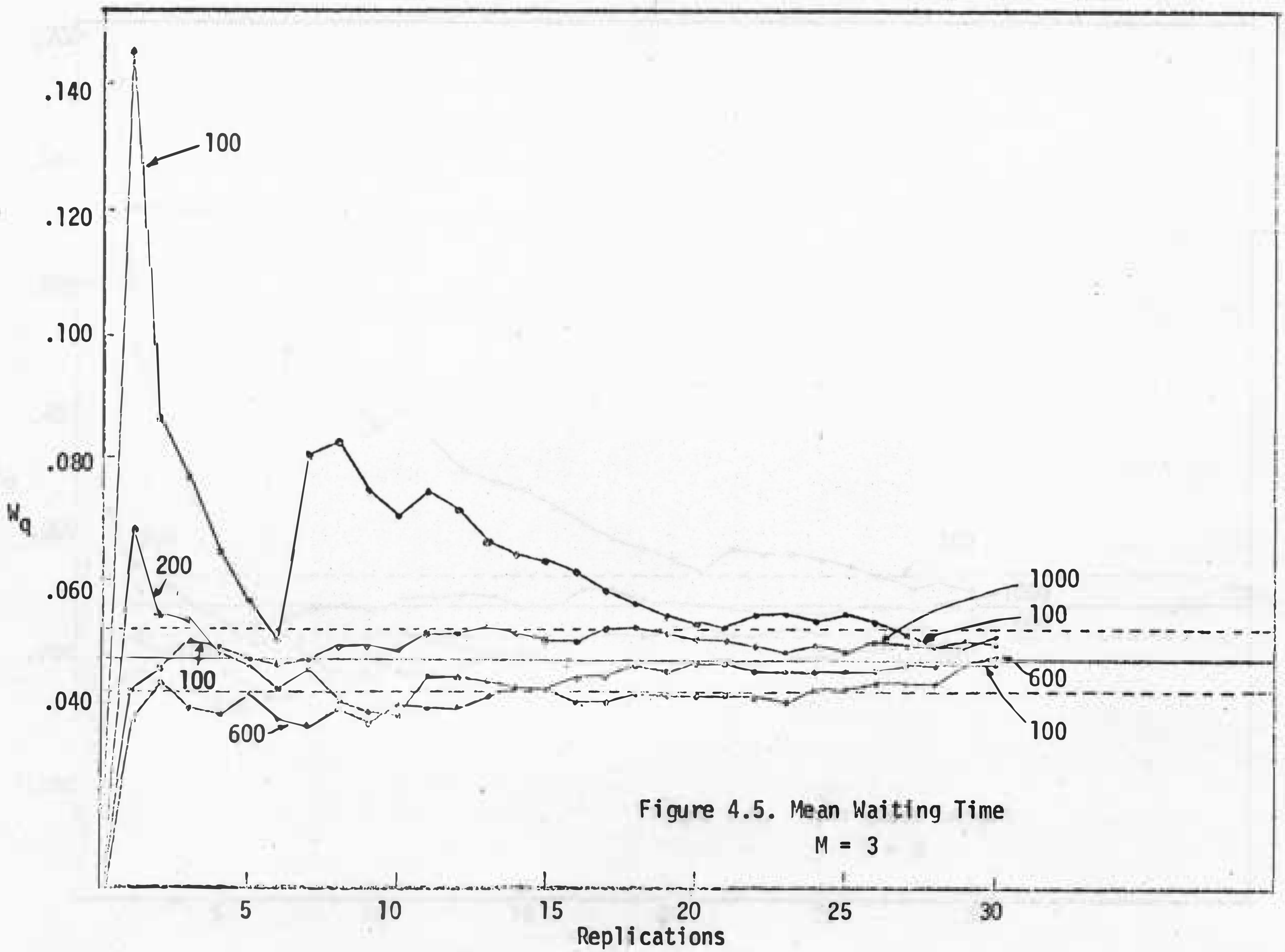

Figure 4.5. Mean Waiting Time
M = 3

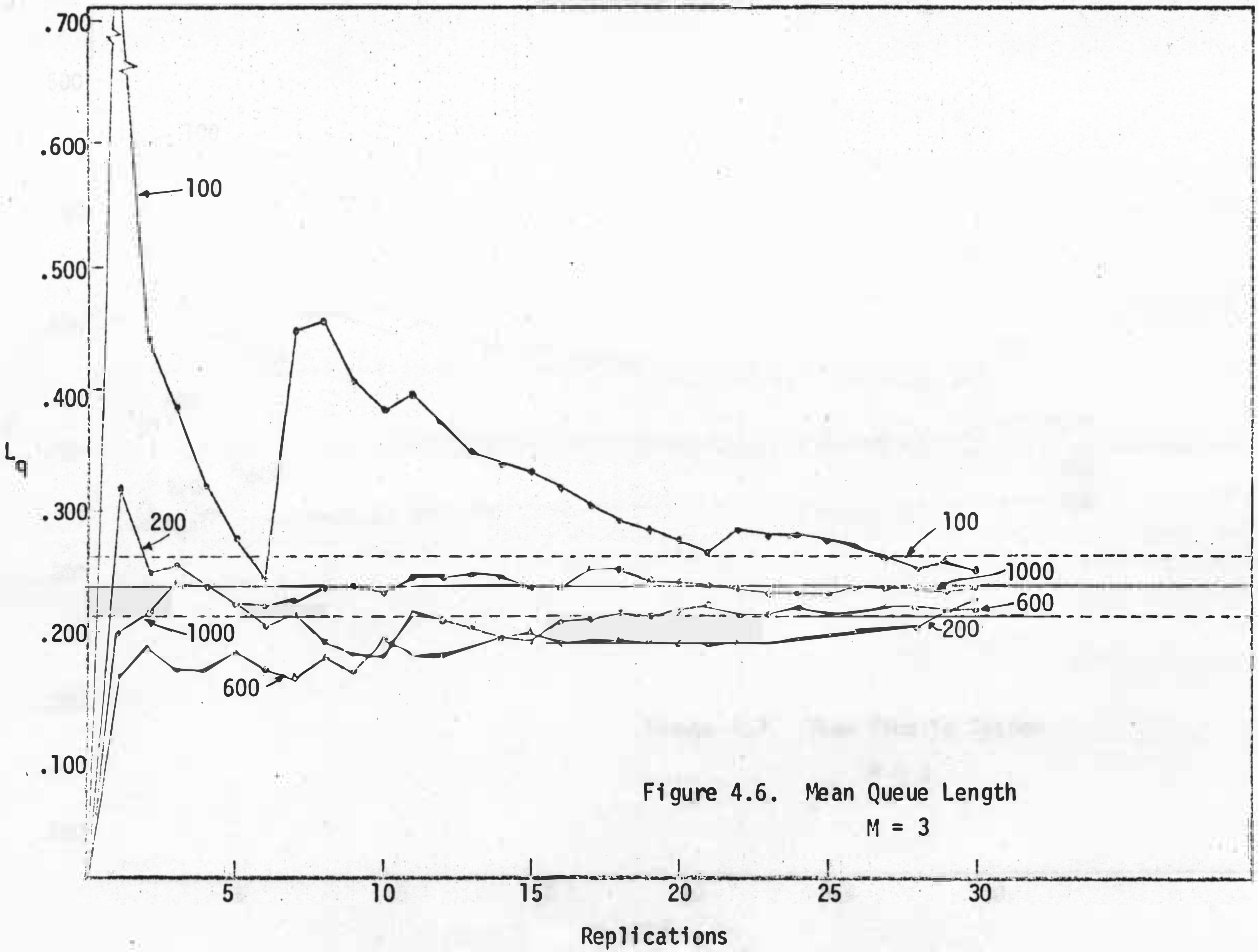

Figure 4.6. Mean Queue Length
M = 3

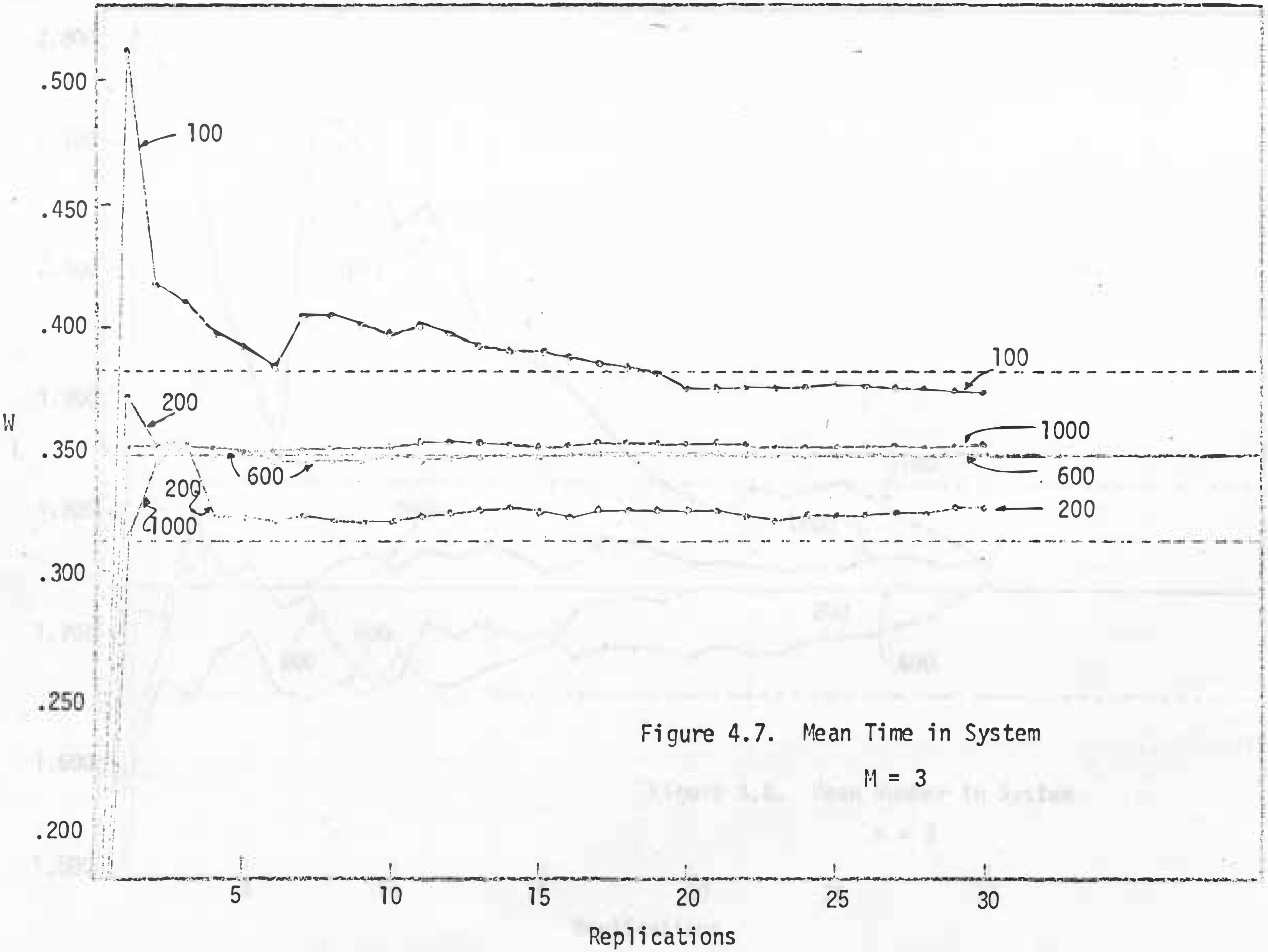

Figure 4.7. Mean Time in System

M = 3

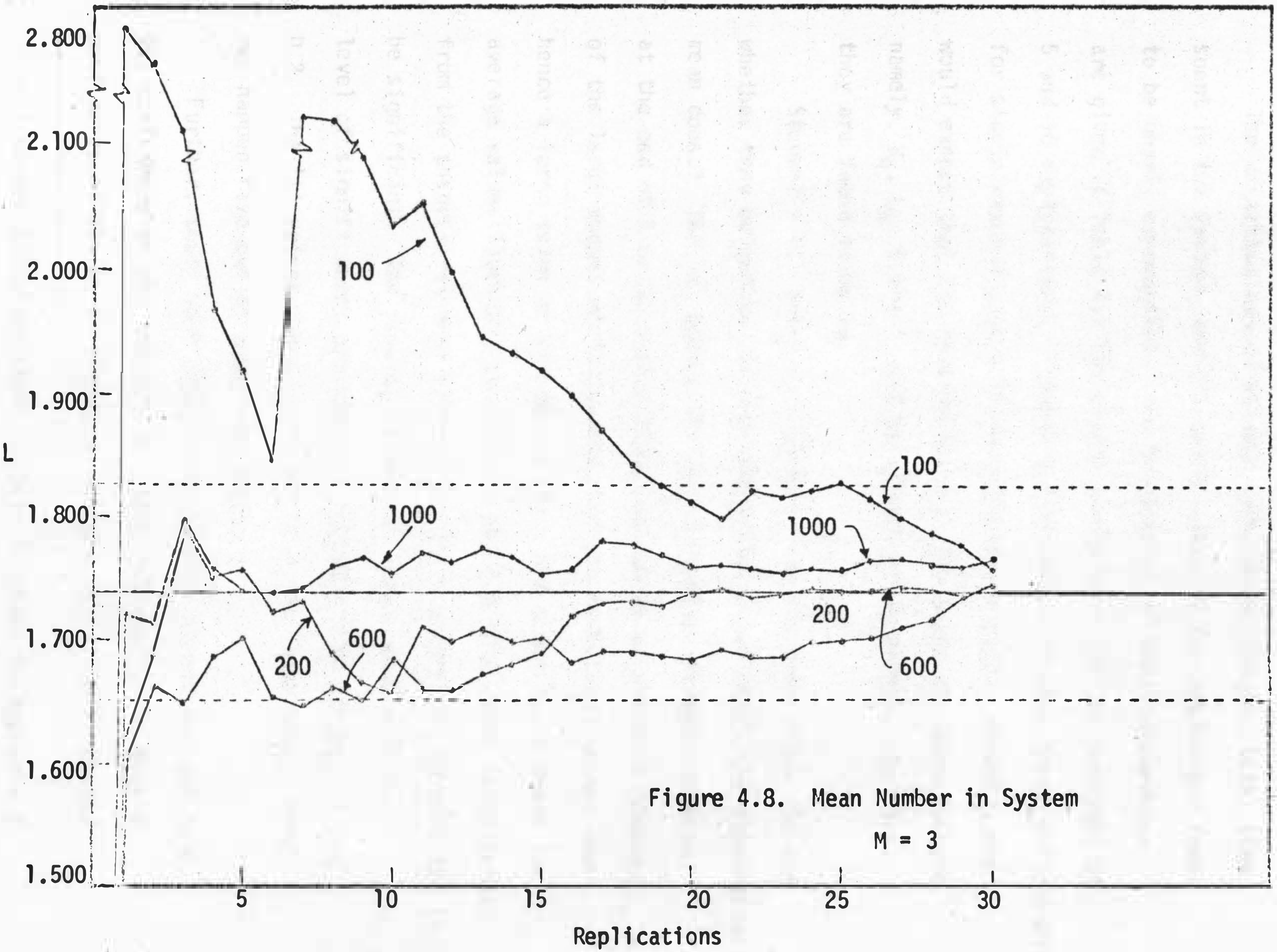

Figure 4.8. Mean Number in System

M = 3

The distributions of waiting time, queue length, total time spent in the system, and the total number in the system are found to be nearly exponential. The frequencies of their occurrence are given in Table 4.2 for single replication and the averages of 5 and 10 replications. Figures 4.9 through 4.12 show their distribution for single channel case with one replication only. However, one would expect that the distributions of the means of these factors, namely, W_q, L_q, W and L will be approximately normal. In fact, they are found to be so.

Student's 't' tests were conducted on the mean values to see whether they belong to the same distribution for which the theoretical mean does.* The 't' values for the cumulative averages obtained at the end of 5 or 10 replications may not be significant because of the large amount of fluctuations of the individual values and hence a large value of standard error. For those cases where the average values fluctuate very little about a line which is different from the theoretical mean value line, the t values can be expected to be significant. The results of the 't' tests, made with 0.05 level of significance, are given in Appendix D in Tables D.1 and D.2. The 't' values are significant in a few cases only. This may happen five out of a hundred times.

Further, tests were made to see whether or not one can have 95% confidence on the accuracy of these estimates. To have a confidence of 95% on a cumulative average, the 1.96 σ spread

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*A sample calculation of 't' test is given in Appendix D.

Table 4.2A

Frequency of occurrence of waiting time

M	ρ	n	.00-0.09	.10-.19	.20-0.29	.30-0.39	.40-0.49	.50-0.59
1	0.5	1	1622	170	135	58	34	22
		5	1455	204	120	83	40	26
		10	1501	196	124	75	39	25
3	0.5	1	1811	88	41	31	22	7
		5	1776	81	53	32	24	10
		10	1770	84	55	31	21	9

Table 4.2B

Frequency of occurrence of queue length

M	ρ	n	0	1	2	3	4	5
1	0.5	1	1593	252	99	62	32	13
		5	1430	244	127	72	42	22
		10	1479	238	126	65	18	6
3	0.5	1	1809	97	57	23	16	6
		5	1746	107	61	26	16	5
		10	1730	107	61	26	17	6

Table 4.2C

Frequency of occurrence of total time spent in the system

M	ρ	n	0.00-0.19	0.20-0.39	0.40-0.59	0.60-0.79	0.80-0.99	1.00-1.19
1	0.5	1	1327	509	141	66	17	1
		5	1186	489	178	78	21	10
		10	1222	482	172	71	23	11
3	0.5	1	881	519	252	169	89	54
		5	814	536	284	168	88	45
		10	817	518	293	171	90	45

Table 4.2D

Frequency of occurrence of total units in the system

M	ρ	n	0	1	2	3	4	5
1	0.5	1	1056	537	252	99	62	32
		5	927	523	244	127	72	42
		10	943	546	238	126	65	18
3	0.5	1	447	687	461	214	97	57
		5	414	634	453	265	107	61
		10	404	628	468	260	107	61

Figure 4.9. Distribution of Waiting Time

M = 1

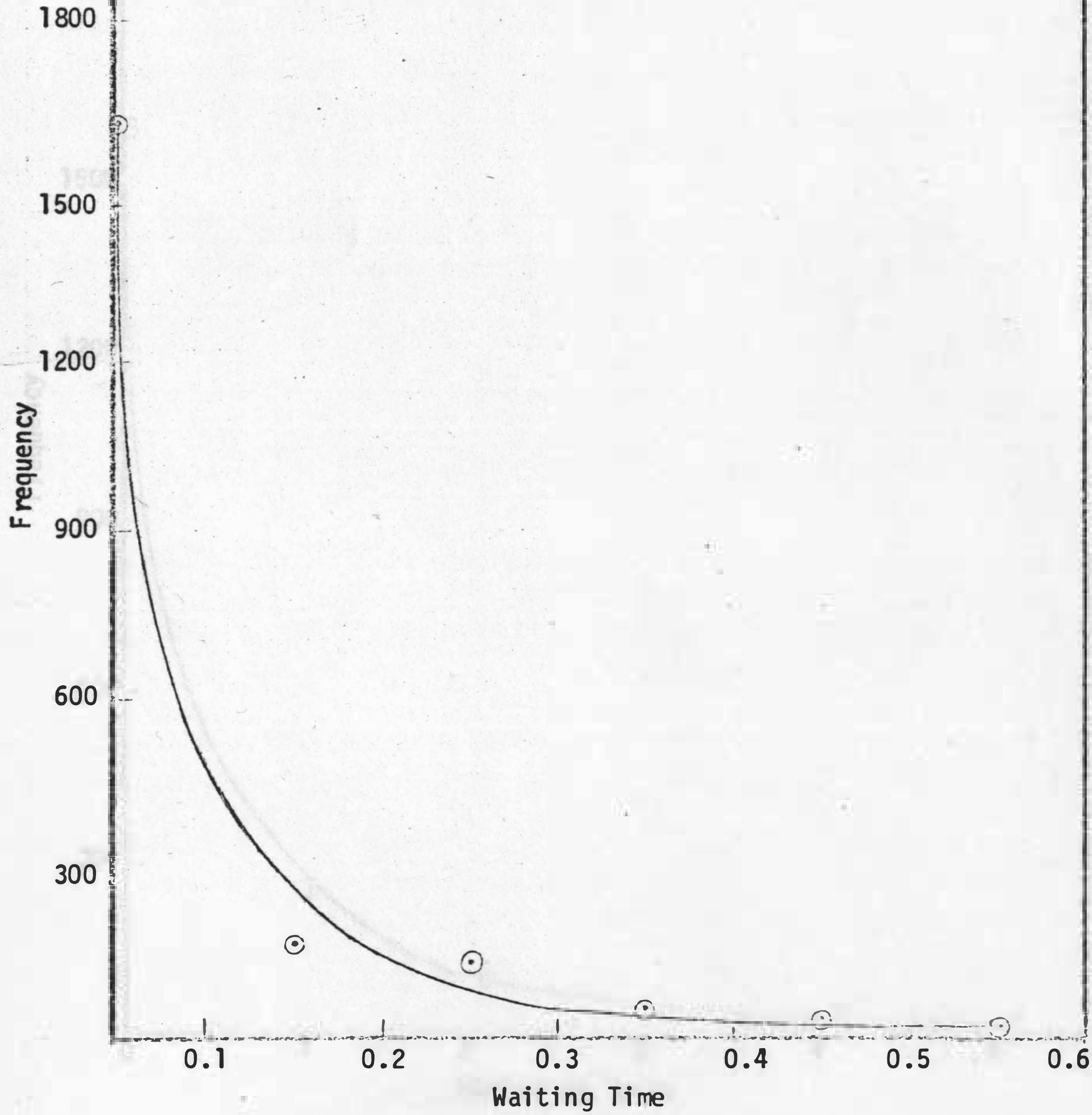


Figure 4.10. Distribution of the Number of Units in Queue

M = 1

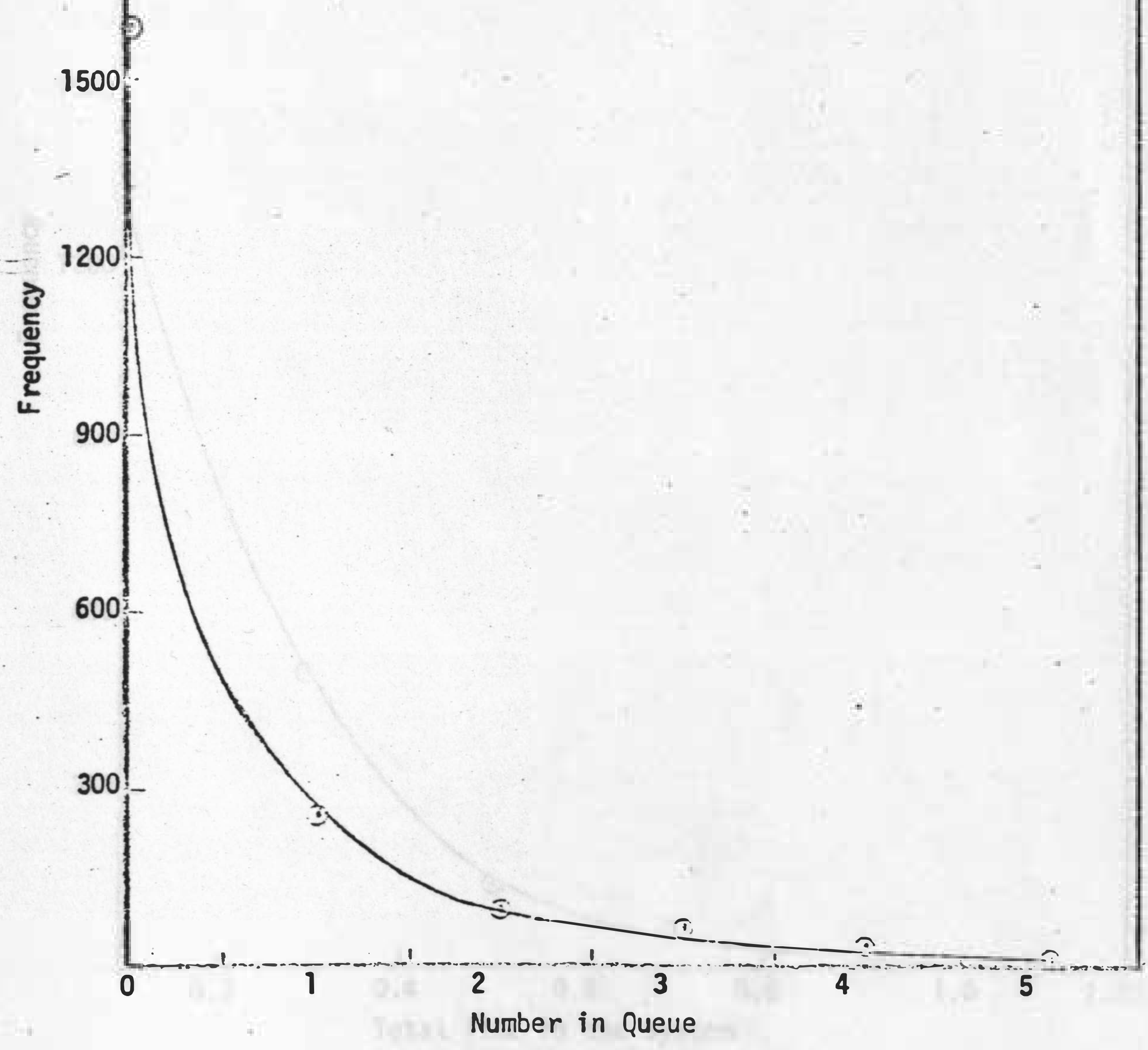


Figure 4.11. Distribution of the Total Time in the System.

M = 1

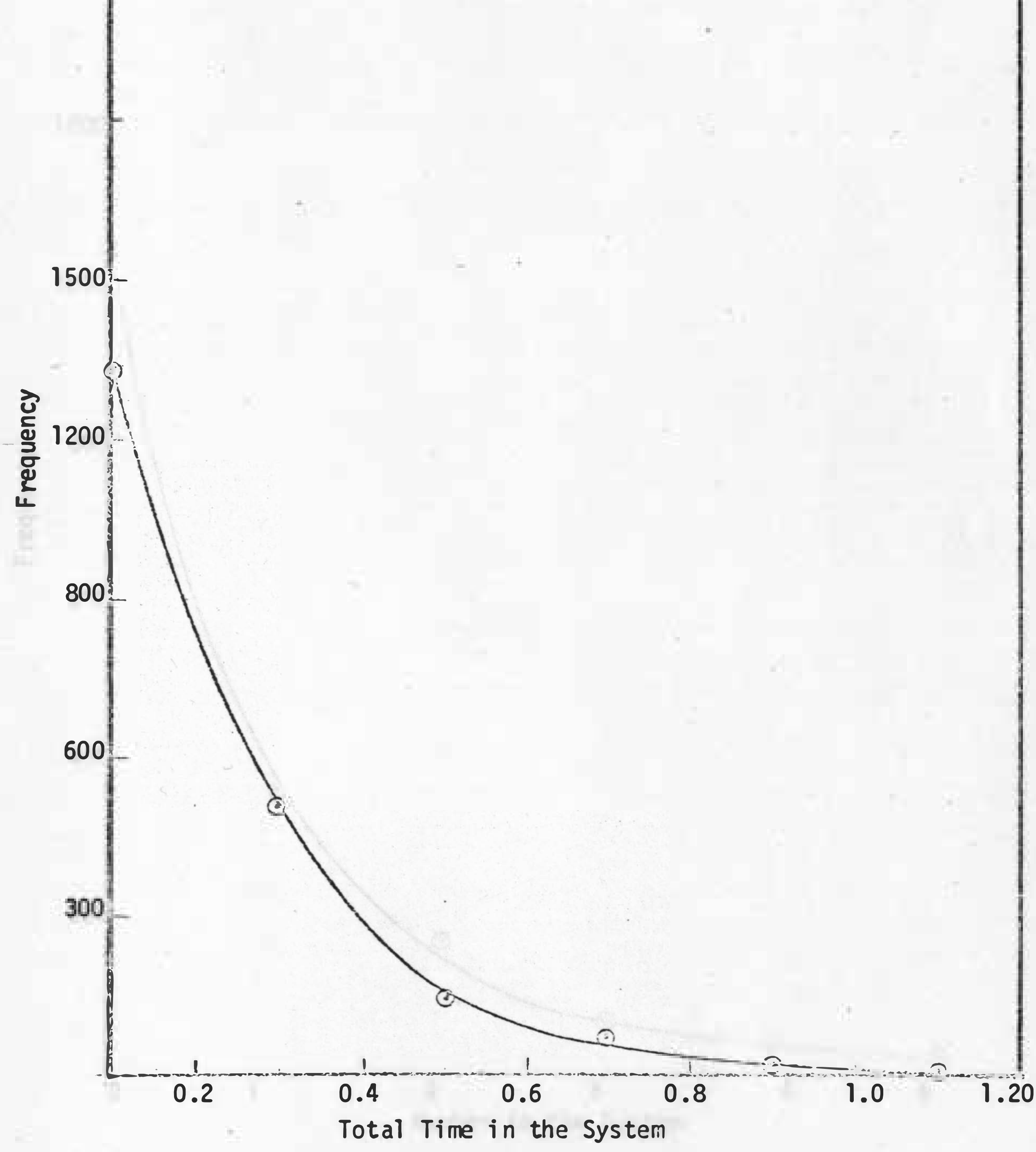
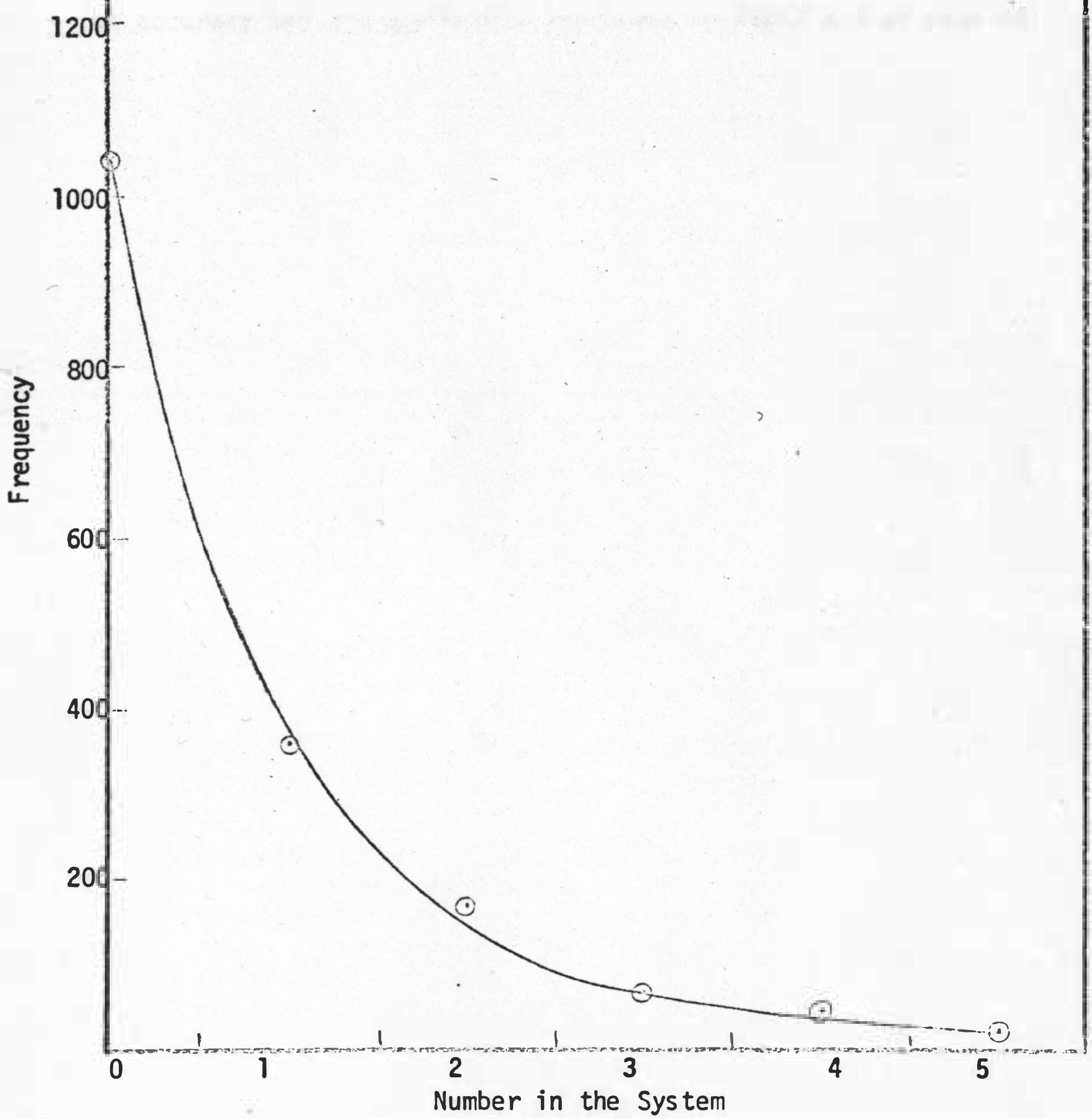


Figure 4.12. Distribution of the Number of Units in the System

M = 1



of its distribution should be within the lower and upper limits. The results of this test are given in Appendix D in Tables D.3 and D.4 on pages 72, and 73.

The final results giving the various combinations of the sample size-replication required to make an estimate of the statistic with $\pm$ 10% accuracy and 95% confidence are given in Table 4.3 on page 48.

Table 4.3

Minimum number of replications required to give an estimate with $\pm$ 10% accuracy and 95% confidence

M=1 ρ =0.5

Statistic	100	200	400	600	800	1000
W_q	> 30	> 30	25	> 30	> 30	25
L_q	> 30	> 30	> 30	> 30	5	20
W	> 30	> 30	15	10	5	5
L	> 30	> 30	15	20	5	5

M=3 ρ =0.5

Statistic	100	200	400	600	800	1000
W_q	> 30	> 30	> 30	> 30	> 30	> 30
L_q	> 30	> 30	> 30	> 30	> 30	> 30
W	> 30	5	5	5	5	5
L	> 30	5	5	5	5	5

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

The following conclusions may be drawn:

1. Replications reduce the number of total units to be simulated.

It can be seen from Figure 2.1 that 5400 units are required to estimate the mean number in the system for the single exponential channel with ρ =0.5. Table 4.3 shows that the minimum number of units required to estimate this statistic with the same degree of accuracy and confidence is 4000, if 5 replications are made each of sample size 800.

2. Choice of statistics affects the number of replications required.

Different statistics require different sample size-replication combinations to obtain the estimates with same degree of accuracy and confidence. For example, W_q requires 25 replications each with a sample size of 400 whereas W requires only 15 replications with the same sample size.

3. Number of channels also affects the number of replications required.

In the case of single channel situation, it requires 25 replications of 1000 units each to estimate W_q. Whereas in the case of 3 parallel channels situation, it requires more than 30 replications of 1000 units each. To estimate W in the single channel case, it is necessary to make more than 30 replications of 200 units each but only 5 replications of the same sample size are required for the 3 parallel channels case.

The following recommendations are made:

1. Elimination of transiency will reduce the required number of replications.

Since the transciency is included in all of these runs, it takes longer time for the mean value to settle down. Mathematical relations have been developed to find out the period of transciency for a few queueing situations. In the case, for which such equations are not available, the transiency may be elimiated by starting the computation of the statistics after a number of arrivals generated. This, of course, will not reduce the total simulation time, except as it reduces the initial fluctuations.

2. Research should be continued for other values of the utilization factor and the number of channels, for the same exponential interarrival time distribution and exponential service time distribution and for other distributions of interarrival times and service times.
3. Different sampling techniques developed and described by Hammersely and Handscomb [5] may be investigated to determine how much they reduce the sample size still further.

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APPENDIX A

A Partial List of the Terms used in Queueing Theory and Simulation*

Event. A noteworthy occurrence in the system is termed as an event. A new unit entering the system and a unit entering the service facility from the queue are two examples of events.

Facility. A facility is a generalized service station, and may be used to indicate a complex of service stations. Service stations and facility are interchangeable terms where the facility is only a single service station.

Parameters. Parameters are the factors such as the mean arrival rate and the mean service rate which define a system.

Pseudorandom Numbers. A sequence of numbers is generated, usually through a computer program, by a rule that permits reproducibility in such a way that any reasonable statistical test will show no significant departure from randomness. Such a generated sequence of numbers are pseudorandom numbers.

Queue. The queue consists of all units that are waiting to be serviced at a specified service station

Queue discipline. Queue discipline is the manner in which the units in the queue are taken into the service facility.

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*Part of this list is from Reference 1.

<u>Service station.</u> The service which is to be performed is accomplished in a service station.

<u>Statistic.</u> Mean waiting time (W_q), mean queue length (L_q), mean time in the system (W) and mean number of units in the system (L) are statistics.

<u>System.</u> The system includes the queue and the service station or stations.

<u>Unit.</u> That which is to be serviced is called a unit.

<u>Utilization factor (ρ).</u> The ratio of the rate of arrival of units to the system rate of service is termed as the utilization factor.

Appendix B

Calculation of the statistics

Single channel case:

λ = 5/time unit $\qquad$ μ =10/time unit $\qquad$ ρ =0.5

Mean waiting time: $W_q = \dfrac{\lambda}{\mu(\mu-\lambda)}$

$= \dfrac{5}{10(10-5)}$

$= \underline{\underline{0.100}}$ time units.

Mean queue length: $L_q = \lambda W_q$

$= 5 \times 0.1$

$= \underline{\underline{0.500}}$

Total time in the system: $W = \dfrac{1}{\mu-\lambda}$

$= \dfrac{1}{(10-5)}$

$= \underline{\underline{0.200}}$ time units.

Total number in the system: $L = \lambda W$

$= 5 \times 0.200$

$= \underline{\underline{1.000}}$

Three Parallel Channel case:

M=3 λ =5/time μ =3.333/time unit

Mean queue length: $L_q = \frac{\rho}{1-\rho} \frac{e_M(\rho M)}{D_{M-1}(\rho M)}$

$$= \frac{0.5}{(1-0.5)} \quad \frac{.1255}{.5299}$$

$$= \underline{\underline{0.237}}$$

Mean waiting time: $W_q = \frac{L_q}{\lambda}$

$$= \frac{0.237}{5}$$

$$= \underline{\underline{0.047}}$$

Total number in the system: $L = L_q + \rho M$

$$= 0.237 + 0.5 \times 3$$

$$= \underline{\underline{1.737}}$$

Total time in the system: $W = \frac{L}{\lambda}$

$$= \frac{1.737}{5}$$

$$= \underline{\underline{0.347}}$$

*The values of $e_M(\rho M)$ and $D_{M-1}(\rho M)$ are obtained from Table V, "Queues, Inventories, and Maintenance", by Philip M. Morse.

Table C.1

Results of the simulation runs

$M = 1$ $\rho = 0.5$ $N = 100$

No.	W_q	CW_q	L_q	CL_q	W	CW	L	CL
1.	0.049	0.049	0.197	0.197	0.152	0.152	0.645	0.645
2.	0.075	0.062	0.391	0.294	0.181	0.166	0.967	0.806
3.	0.160	0.094	0.816	0.468	0.261	0.198	1.374	0.995
4.	0.042	0.081	0.156	0.390	0.143	0.184	0.539	0.881
5.	0.096	0.084	0.476	0.407	0.198	0.187	1.031	0.911
6.	0.131	0.092	0.604	0.440	0.245	0.197	1.152	0.951
7.	0.105	0.094	0.550	0.456	0.226	0.201	1.211	0.988
8.	0.053	0.089	0.279	0.434	0.146	0.194	0.779	0.962
9.	0.055	0.085	0.294	0.418	0.149	0.189	0.786	0.943
10.	0.045	0.081	0.208	0.397	0.140	0.184	0.675	0.916
11.	0.113	0.084	0.574	0.413	0.230	0.188	1.105	0.933
12.	0.116	0.087	0.617	0.430	0.210	0.190	1.166	0.953
13.	0.356	0.107	2.047	0.555	0.466	0.211	2.667	1.084
14.	0.078	0.105	0.349	0.540	0.174	0.209	0.880	1.070
15.	0.171	0.110	1.124	0.579	0.267	0.212	1.745	1.115
16.	0.076	0.107	0.464	0.572	0.174	0.210	1.072	1.112
17.	0.070	0.105	0.335	0.558	0.163	0.207	0.827	1.095
18.	0.183	0.110	1.073	0.586	0.286	0.212	1.701	1.129
19.	0.094	0.109	0.450	0.579	0.200	0.211	1.000	1.122
20.	0.122	0.109	0.676	0.584	0.238	0.212	1.365	1.134
21.	0.194	0.113	1.054	0.606	0.297	0.217	1.643	1.159
22.	0.099	0.113	0.327	0.594	0.216	0.216	0.802	1.142
23.	0.069	0.111	0.304	0.581	0.173	0.215	0.807	1.128
24.	0.144	0.112	0.818	0.591	0.253	0.216	1.392	1.139
25.	0.061	0.110	0.294	0.579	0.161	0.214	0.743	1.123
26.	0.093	0.110	0.385	0.572	0.198	0.213	0.873	1.113
27.	0.114	0.110	0.478	0.568	0.235	0.214	1.048	1.111
28.	0.063	0.108	0.307	0.559	0.164	0.212	0.891	1.103
29.	0.064	0.107	0.316	0.551	0.168	0.211	0.805	1.093
30.	0.129	0.107	0.680	0.555	0.213	0.211	1.171	1.095

Table C.2

Results of simulation runs

M = 1 ρ = 0.5 N = 200

No.	W_q	CW_q	L_q	CL_q	W	CW	L	CL
1.	0.119	0.119	0.670	0.670	0.225	0.225	1.327	1.327
2.	0.138	0.128	0.635	0.652	0.246	0.235	1.141	1.234
3.	0.153	0.137	0.806	0.704	0.262	0.244	1.398	1.289
4.	0.097	0.127	0.510	0.655	0.187	0.230	0.997	1.216
5.	0.063	0.114	0.343	0.593	0.154	0.215	0.826	1.138
6.	0.059	0.105	0.261	0.538	0.155	0.205	0.693	1.064
7.	0.064	0.099	0.301	0.504	0.162	0.199	0.799	1.026
8.	0.085	0.097	0.379	0.488	0.178	0.196	0.871	1.007
9.	0.055	0.093	0.239	0.461	0.164	0.192	0.756	0.979
10.	0.055	0.089	0.275	0.442	0.148	0.188	0.727	0.954
11.	0.117	0.091	0.594	0.456	0.223	0.191	1.170	0.973
12.	0.185	0.099	1.022	0.503	0.292	0.200	1.642	1.029
13.	0.058	0.096	0.288	0.486	0.152	0.194	0.794	1.011
14.	0.063	0.094	0.242	0.469	0.159	0.193	0.669	0.987
15.	0.102	0.094	0.566	0.475	0.204	0.194	1.145	0.997
16.	0.091	0.094	0.392	0.479	0.198	0.194	0.950	0.994
17.	0.063	0.092	0.326	0.462	0.155	0.192	0.785	0.982
18.	0.130	0.094	0.668	0.473	0.227	0.194	1.165	0.992
19.	0.065	0.093	0.318	0.465	0.160	0.192	0.828	0.983
20.	0.111	0.094	0.577	0.471	0.208	0.193	1.064	0.987
21.	0.114	0.095	0.501	0.472	0.223	0.194	1.031	0.989
22.	0.101	0.095	0.459	0.471	0.205	0.195	0.973	0.989
23.	0.076	0.094	0.365	0.467	0.178	0.194	0.881	0.984
24.	0.099	0.094	0.475	0.467	0.205	0.194	1.007	0.985
25.	0.079	0.094	0.361	0.463	0.170	0.193	0.805	0.978
26.	0.079	0.093	0.390	0.460	0.173	0.193	0.868	0.974
27.	0.079	0.093	0.420	0.459	0.170	0.192	0.917	0.972
28.	0.124	0.094	0.652	0.466	0.228	0.193	1.250	0.981
29.	0.070	0.093	0.296	0.460	0.178	0.193	0.816	0.976
30.	0.083	0.093	0.456	0.460	0.183	0.192	0.992	0.976

Table C.3

Results of simulation runs

M = 1 $\rho = 0.5$ N = 400

No.	W_q	CW_q	L_q	CL_q	W	CW	L	CL
1.	0.108	0.108	0.502	0.502	0.215	0.215	1.055	1.055
2.	0.148	0.128	0.737	0.619	0.246	0.231	1.238	1.146
3.	0.104	0.120	0.550	0.596	0.203	0.221	1.092	1.128
4.	0.092	0.113	0.463	0.563	0.190	0.213	0.986	1.093
5.	0.099	0.110	0.484	0.547	0.197	0.210	1.000	1.074
6.	0.111	0.110	0.453	0.531	0.225	0.213	0.979	1.058
7.	0.093	0.108	0.451	0.521	0.194	0.210	1.000	1.050
8.	0.111	0.108	0.553	0.525	0.215	0.211	1.095	1.056
9.	0.103	0.108	0.458	0.517	0.205	0.210	0.933	1.042
10.	0.112	0.108	0.590	0.525	0.210	0.210	1.145	1.052
11.	0.138	0.111	0.697	0.540	0.243	0.213	1.270	1.072
12.	0.075	0.108	0.369	0.526	0.169	0.209	0.871	1.055
13.	0.071	0.105	0.330	0.511	0.168	0.206	0.791	1.035
14.	0.066	0.102	0.295	0.496	0.160	0.203	0.771	1.016
15.	0.106	0.103	0.526	0.498	0.211	0.203	1.081	1.020
16.	0.124	0.104	0.563	0.502	0.235	0.205	1.092	1.025
17.	0.105	0.104	0.527	0.503	0.203	0.205	1.066	1.027
18.	0.113	0.104	0.534	0.505	0.219	0.206	1.056	1.029
19.	0.079	0.103	0.394	0.499	0.179	0.205	0.915	1.023
20.	0.089	0.102	0.435	0.496	0.181	0.203	0.915	1.017
21.	0.111	0.103	0.530	0.497	0.214	0.204	1.045	1.019
22.	0.081	0.102	0.394	0.493	0.179	0.203	0.922	1.014
23.	0.094	0.101	0.484	0.492	0.198	0.203	1.038	1.015
24.	0.088	0.101	0.384	0.488	0.189	0.202	0.855	1.009
25.	0.052	0.099	0.244	0.478	0.143	0.200	0.697	0.996
26.	0.177	0.102	0.964	0.497	0.283	0.203	1.574	1.018
27.	0.053	0.100	0.247	0.488	0.147	0.201	0.707	1.007
28.	0.094	0.100	0.439	0.486	0.192	0.200	0.960	1.005
29.	0.088	0.100	0.393	0.483	0.186	0.200	0.882	1.001
30.	0.080	0.099	0.336	0.478	0.183	0.199	0.816	0.995

Table C.4

Results of simulation runs

M = 1 ρ = 0.5 N = 600

No.	W_q	CW_q	L_q	CL_q	W	CW	L	CL
1.	0.077	0.077	0.346	0.346	0.174	0.174	0.816	0.816
2.	0.080	0.078	0.353	0.349	0.180	0.177	0.858	0.837
3.	0.085	0.080	0.385	0.361	0.184	0.179	0.881	0.852
4.	0.087	0.082	0.416	0.375	0.186	0.181	0.926	0.870
5.	0.089	0.083	0.436	0.387	0.186	0.182	0.929	0.882
6.	0.102	0.087	0.552	0.415	0.200	0.185	1.111	0.920
7.	0.117	0.091	0.553	0.434	0.221	0.190	1.076	0.942
8.	0.102	0.092	0.495	0.442	0.199	0.191	0.999	0.950
9.	0.123	0.096	0.620	0.462	0.224	0.195	1.171	0.974
10.	0.084	0.095	0.407	0.436	0.184	0.194	0.914	0.968
11.	0.088	0.094	0.437	0.455	0.187	0.193	0.942	0.960
12.	0.080	0.093	0.367	0.447	0.180	0.192	0.872	0.958
13.	0.120	0.095	0.588	0.458	0.221	0.194	1.122	0.970
14.	0.069	0.093	0.339	0.450	0.165	0.192	0.831	0.961
15.	0.084	0.092	0.410	0.447	0.181	0.192	0.890	0.956
16.	0.090	0.092	0.454	0.447	0.187	0.191	0.955	0.956
17.	0.089	0.092	0.423	0.446	0.186	0.191	0.931	0.954
18.	0.105	0.093	0.530	0.451	0.205	0.192	1.077	0.961
19.	0.086	0.092	0.415	0.449	0.182	0.191	0.907	0.958
20.	0.091	0.092	0.432	0.448	0.192	0.191	0.956	0.958
21.	0.083	0.092	0.425	0.447	0.180	0.191	0.946	0.958
22.	0.083	0.091	0.396	0.444	0.184	0.190	0.924	0.956
23.	0.065	0.090	0.289	0.438	0.159	0.189	0.762	0.948
24.	0.097	0.091	0.469	0.439	0.198	0.189	0.990	0.949
25.	0.110	0.091	0.556	0.444	0.211	0.190	1.094	0.955
26.	0.127	0.093	0.641	0.451	0.235	0.192	1.209	0.965
27.	0.093	0.093	0.464	0.452	0.194	0.192	0.976	0.965
28.	0.088	0.093	0.299	0.450	0.190	0.192	0.939	0.964
29.	0.078	0.092	0.408	0.448	0.168	0.191	0.904	0.962
30.	0.063	0.091	0.286	0.443	0.163	0.190	0.798	0.957

Table C.5

Results of simulation runs

M = 1 ρ = 0.5 N = 800

No.	W_q	CW_q	L_q	CL_q	W	CW	L	CL
1.	0.099	0.099	0.486	0.486	0.199	0.199	1.010	1.010
2.	0.111	0.105	0.544	0.515	0.211	0.205	1.085	1.047
3.	0.105	0.105	0.498	0.510	0.207	0.206	1.027	1.041
4.	0.097	0.103	0.450	0.495	0.199	0.204	0.957	1.020
5.	0.119	0.106	0.523	0.500	0.223	0.208	1.012	1.018
6.	0.075	0.101	0.351	0.475	0.171	0.202	0.837	0.988
7.	0.083	0.099	0.417	0.467	0.180	0.199	0.915	0.978
8.	0.105	0.099	0.521	0.474	0.199	0.199	1.012	0.982
9.	0.094	0.099	0.437	0.470	0.195	0.198	0.952	0.979
10.	0.123	0.101	0.615	0.484	0.227	0.201	1.196	1.000
11.	0.096	0.101	0.479	0.484	0.193	0.200	1.018	1.002
12.	0.104	0.101	0.486	0.484	0.207	0.201	1.012	1.003
13.	0.091	0.100	0.431	0.480	0.189	0.200	0.932	0.997
14.	0.103	0.100	0.486	0.480	0.201	0.200	0.983	0.996
15.	0.119	0.102	0.586	0.487	0.219	0.201	1.138	1.006
16.	0.101	0.102	0.496	0.488	0.196	0.201	1.018	1.007
17.	0.089	0.101	0.433	0.485	0.192	0.201	0.976	1.005
18.	0.119	0.102	0.643	0.493	0.216	0.201	1.192	1.015
19.	0.082	0.101	0.397	0.488	0.175	0.200	0.879	1.008
20.	0.126	0.102	0.658	0.497	0.230	0.201	1.255	1.020
21.	0.091	0.102	0.424	0.493	0.191	0.201	0.919	1.016
22.	0.088	0.101	0.407	0.489	0.190	0.200	0.916	1.011
23.	0.101	0.101	0.445	0.488	0.200	0.200	0.934	1.008
24.	0.083	0.100	0.392	0.484	0.178	0.199	0.886	1.003
25.	0.128	0.101	0.602	0.488	0.233	0.201	1.129	1.008
26.	0.096	0.101	0.490	0.488	0.189	0.200	1.004	1.008
27.	0.074	0.100	0.359	0.484	0.171	0.199	0.878	1.003
28.	0.104	0.100	0.523	0.483	0.207	0.200	1.073	1.005
29.	0.081	0.100	0.392	0.482	0.180	0.199	0.910	1.002
30.	0.087	0.099	0.435	0.480	0.186	0.198	0.952	1.000

Table C.6

Results of simulation runs

$M = 1$ $\rho = 0.5$ $N = 1000$

No.	W_q	CW_q	L_q	CL_q	W	CW	L	CL
1.	0.102	0.102	0.491	0.491	0.199	0.199	1.007	1.007
2.	0.099	0.100	0.487	0.489	0.198	0.198	1.025	1.016
3.	0.084	0.095	0.385	0.454	0.188	0.195	0.907	0.980
4.	0.101	0.096	0.496	0.465	0.203	0.197	1.039	0.994
5.	0.088	0.095	0.453	0.462	0.189	0.195	0.991	0.994
6.	0.081	0.092	0.372	0.447	0.175	0.192	0.837	0.968
7.	0.065	0.089	0.298	0.426	0.166	0.188	0.784	0.941
8.	0.152	0.096	0.760	0.468	0.255	0.197	1.305	0.987
9.	0.104	0.097	0.486	0.470	0.208	0.198	1.019	0.990
10.	0.108	0.098	0.541	0.477	0.207	0.199	1.066	0.998
11.	0.087	0.097	0.441	0.474	0.185	0.198	0.974	0.996
12.	0.121	0.099	0.594	0.484	0.224	0.200	1.130	1.007
13.	0.105	0.100	0.499	0.485	0.203	0.200	1.014	1.007
14.	0.115	0.101	0.549	0.489	0.212	0.201	1.042	1.010
15.	0.107	0.101	0.561	0.494	0.209	0.201	1.124	1.018
16.	0.119	0.102	0.579	0.500	0.218	0.203	1.091	1.022
17.	0.104	0.103	0.536	0.502	0.201	0.202	1.054	1.024
18.	0.096	0.102	0.473	0.500	0.193	0.202	0.974	1.021
19.	0.084	0.101	0.399	0.495	0.183	0.201	0.905	1.015
20.	0.126	0.102	0.608	0.500	0.226	0.202	1.132	1.021
21.	0.088	0.103	0.404	0.496	0.187	0.201	0.892	1.015
22.	0.104	0.102	0.498	0.496	0.208	0.202	1.029	1.015
23.	0.111	0.102	0.568	0.499	0.211	0.202	1.111	1.020
24.	0.074	0.101	0.335	0.492	0.173	0.201	0.821	1.011
25.	0.112	0.102	0.525	0.494	0.218	0.202	1.066	1.014
26.	0.118	0.102	0.557	0.496	0.222	0.202	1.012	1.017
27.	0.120	0.103	0.604	0.500	0.220	0.203	1.143	1.022
28.	0.130	0.104	0.636	0.505	0.230	0.204	1.157	1.026
29.	0.101	0.104	0.521	0.505	0.200	0.204	1.052	1.027
30.	0.098	0.103	0.473	0.504	0.196	0.204	0.974	1.026

Table C.7

Results of simulation runs

M = 3 ρ = 0.5 N = 100

No.	W_q	CW_q	L_q	CL_q	W	CW	L	CL
1.	0.145	0.145	0.746	0.746	0.523	0.523	2.778	2.778
2.	0.027	0.086	0.132	0.439	0.354	0.439	1.721	2.255
3.	0.056	0.076	0.276	0.384	0.384	0.421	1.903	2.137
4.	0.029	0.064	0.129	0.321	0.314	0.394	1.448	1.965
5.	0.023	0.056	0.104	0.277	0.350	0.385	1.724	1.917
6.	0.019	0.050	0.079	0.244	0.278	0.367	1.500	1.847
7.	0.260	0.080	1.662	0.447	0.657	0.409	4.175	2.180
8.	0.101	0.082	0.510	0.455	0.410	0.409	2.063	2.165
9.	0.010	0.074	0.043	0.409	0.345	0.402	1.463	2.087
10.	0.033	0.070	0.139	0.382	0.343	0.396	1.537	2.032
11.	0.114	0.074	0.530	0.395	0.460	0.402	2.223	2.050
12.	0.030	0.071	0.127	0.373	0.326	0.395	1.465	2.001
13.	0.010	0.066	0.054	0.348	0.261	0.385	1.249	1.943
14.	0.034	0.064	0.200	0.338	0.301	0.379	1.749	1.932
15.	0.057	0.063	0.236	0.331	0.363	0.378	1.697	1.917
16.	0.034	0.061	0.157	0.320	0.335	0.375	1.645	1.900
17.	0.008	0.058	0.048	0.304	0.286	0.370	1.365	1.868
18.	0.017	0.056	0.060	0.291	0.313	0.367	1.336	1.839
19.	0.027	0.054	0.142	0.283	0.289	0.363	1.563	1.824
20.	0.023	0.053	0.116	0.274	0.325	0.361	1.608	1.813
21.	0.024	0.052	0.098	0.266	0.336	0.360	1.482	1.798
22.	0.112	0.054	0.633	0.283	0.384	0.361	2.241	1.818
23.	0.036	0.054	0.173	0.278	0.355	0.361	1.751	1.815
24.	0.052	0.053	0.246	0.277	0.389	0.362	1.933	1.820
25.	0.055	0.054	0.240	0.275	0.414	0.364	1.990	1.827
26.	0.028	0.053	0.121	0.269	0.316	0.362	1.500	1.814
27.	0.011	0.051	0.031	0.260	0.334	0.361	1.376	1.798
28.	0.008	0.049	0.033	0.252	0.294	0.359	1.371	1.783
29.	0.077	0.050	0.340	0.255	0.336	0.358	1.549	1.774
30.	0.018	0.049	0.057	0.249	0.323	0.357	1.188	1.755

Table C.8

Results of simulation runs

M = 3 ρ = 0.5 N = 200

No.	W_q	CW_q	L_q	CL_q	W	CW	L	CL
1.	0.068	0.068	0.318	0.318	0.368	0.368	1.718	1.718
2.	0.040	0.054	0.181	0.249	0.339	0.353	1.707	1.712
3.	0.051	0.053	0.264	0.254	0.360	0.355	1.935	1.787
4.	0.034	0.048	0.181	0.236	0.308	0.344	1.641	1.750
5.	0.036	0.046	0.172	0.223	0.351	0.345	1.773	1.755
6.	0.026	0.042	0.114	0.205	0.321	0.341	1.563	1.723
7	0.062	0.045	0.268	0.214	0.371	0.345	1.759	1.728
8.	0.009	0.040	0.039	0.192	0.307	0.341	1.406	1.688
9.	0.023	0.038	0.098	0.182	0.325	0.339	1.467	1.663
10.	0.038	0.038	0.165	0.180	0.351	0.340	1.587	1.656
11.	0.102	0.044	0.555	0.214	0.404	0.346	2.224	1.707
12.	0.036	0.044	0.140	0.208	0.363	0.347	1.597	1.698
13.	0.031	0.043	0.128	0.202	0.380	0.350	1.806	1.706
14.	0.034	0.042	0.129	0.196	0.363	0.351	1.585	1.698
15.	0.040	0.042	0.207	0.197	0.323	0.349	1.728	1.700
16.	0.018	0.040	0.076	0.190	0.298	0.346	1.392	1.680
17.	0.042	0.040	0.215	0.191	0.326	0.345	1.785	1.687
18.	0.049	0.041	0.224	0.193	0.351	0.345	1.702	1.687
19.	0.035	0.041	0.160	0.191	0.335	0.344	1.615	1.684
20.	0.044	0.041	0.172	0.190	0.365	0.345	1.643	1.682
21.	0.048	0.041	0.208	0.191	0.346	0.346	1.844	1.689
22.	0.033	0.041	0.154	0.189	0.307	0.344	1.580	1.684
23.	0.034	0.040	0.178	0.189	0.318	0.343	1.673	1.684
24.	0.066	0.042	0.311	0.194	0.399	0.345	1.957	1.695
25.	0.046	0.042	0.226	0.195	0.336	0.345	1.745	1.697
26.	0.068	0.043	0.303	0.199	0.380	0.346	1.769	1.700
27.	0.052	0.043	0.249	0.201	0.374	0.347	1.895	1.707
28.	0.041	0.043	0.174	0.200	0.384	0.347	1.862	1.713
29.	0.118	0.046	0.543	0.212	0.478	0.353	2.287	1.733
30.	0.060	0.046	0.316	0.216	0.371	0.353	2.020	1.742

Table C.9

Results of simulation runs

M = 3 ρ = 0.5 N = 400

No.	W_q	CW_q	L_q	CL_q	W	CW	L	CL
1.	0.054	0.054	0.257	0.257	0.353	0.353	1.727	1.727
2.	0.042	0.048	0.223	0.240	0.334	0.343	1.811	1.769
3.	0.030	0.042	0.140	0.207	0.335	0.341	1.673	1.737
4.	0.035	0.040	0.151	0.193	0.337	0.340	1.589	1.700
5.	0.030	0.038	0.135	0.181	0.330	0.338	1.529	1.606
6.	0.073	0.044	0.344	0.208	0.391	0.347	1.921	1.708
7.	0.033	0.042	0.125	0.196	0.368	0.350	1.677	1.704
8.	0.029	0.041	0.132	0.188	0.308	0.345	1.537	1.683
9.	0.045	0.041	0.219	0.192	0.337	0.344	1.741	1.689
10.	0.039	0.041	0.163	0.189	0.349	0.344	1.639	1.684
11.	0.041	0.041	0.181	0.188	0.328	0.343	1.714	1.687
12.	0.050	0.042	0.245	0.193	0.358	0.344	1.833	1.699
13.	0.057	0.043	0.265	0.198	0.356	0.345	1.760	1.704
14.	0.046	0.043	0.216	0.200	0.378	0.347	1.883	1.717
15.	0.068	0.045	0.321	0.208	0.388	0.350	1.929	1.731
16.	0.025	0.044	0.103	0.201	0.320	0.348	1.505	1.717
17.	0.033	0.043	0.162	0.199	0.339	0.347	1.708	1.716
18.	0.027	0.042	0.126	0.195	0.331	0.347	1.607	1.710
19.	0.035	0.042	0.162	0.193	0.332	0.346	1.651	1.707
20.	0.038	0.041	0.172	0.192	0.337	0.345	1.690	1.706
21.	0.026	0.041	0.124	0.189	0.310	0.344	1.539	1.698
22.	0.083	0.043	0.413	0.199	0.410	0.347	2.125	1.718
23.	0.049	0.043	0.237	0.201	0.352	0.347	1.805	1.721
24.	0.039	0.043	0.159	0.199	0.329	0.346	1.572	1.715
25.	0.090	0.045	0.470	0.210	0.396	0.348	2.161	1.733
26.	0.047	0.045	0.232	0.211	0.349	0.348	1.793	1.735
27.	0.046	0.045	0.218	0.211	0.340	0.348	1.682	1.733
28.	0.032	0.044	0.159	0.209	0.317	0.347	1.586	1.728
29.	0.019	0.043	0.080	0.205	0.293	0.345	1.474	1.719
30,	0.030	0.043	0.138	0.202	0.316	0.344	1.535	1.737

Table C.10

Results of simulation runs

M =3 ρ = 0.5 N = 600

No.	W_q	CW_q	L_q	CL_q	W	CW	L	CL
1.	0.037	0.037	0.164	0.164	0.352	0.352	1.604	1.604
2.	0.049	0.043	0.207	0.186	0.358	0.355	1.710	1.657
3.	0.031	0.039	0.134	0.168	0.332	0.347	1.627	1.647
4.	0.034	0.038	0.171	0.169	0.340	0.346	1.796	1.684
5.	0.052	0.041	0.240	0.183	0.358	0.348	1.752	1.698
6.	0.019	0.037	0.087	0.167	0.303	0.341	1.419	1.651
7.	0.029	0.036	0.125	0.161	0.336	0.340	1.616	1.646
8.	0.061	0.039	0.290	0.177	0.354	0.342	1.748	1.659
9.	0.023	0.037	0.098	0.168	0.322	0.339	1.574	1.650
10.	0.080	0.041	0.398	0.191	0.386	0.344	1.984	1.683
11.	0.016	0.039	0.073	0.180	0.307	0.341	1.415	1.659
12.	0.043	0.039	0.183	0.180	0.358	0.342	1.636	1.657
13.	0.058	0.041	0.284	0.188	0.362	0.344	1.834	1.670
14.	0.056	0.042	0.273	0.194	0.347	0.344	1.772	1.678
15.	0.037	0.042	0.181	0.193	0.349	0.344	1.819	1.687
16.	0.082	0.044	0.431	0.208	0.401	0.348	2.183	1.718
17.	0.044	0.044	0.233	0.210	0.349	0.348	1.867	1.727
18.	0.070	0.046	0.312	0.215	0.375	0.349	1.725	1.728
19.	0.036	0.045	0.180	0.213	0.324	0.348	1.688	1.726
20.	0.060	0.046	0.289	0.217	0.374	0.349	1.889	1.734
21.	0.054	0.046	0.256	0.219	0.346	0.349	1.811	1.738
22.	0.028	0.045	0.136	0.215	0.318	0.348	1.614	1.732
23.	0.042	0.045	0.210	0.215	0.343	0.348	1.781	1.734
24.	0.051	0.045	0.251	0.217	0.351	0.348	1.821	1.738
25.	0.039	0.045	0.186	0.215	0.344	0.348	1.699	1.737
26.	0.041	0.045	0.196	0.215	0.353	0.348	1.732	1.736
27.	0.061	0.046	0.308	0.218	0.353	0.348	1.853	1.741
28.	0.050	0.046	0.225	0.218	0.359	0.348	1.698	1.739
29.	0.041	0.046	0.174	0.217	0.338	0.348	1.555	1.733
30.	0.082	0.047	0.426	0.224	0.390	0.349	2.028	1.743

Table C.11

Results of simulation runs

M = 3 ρ = 0.5 N = 800

No.	W_q	CW_q	Lq	CL_q	W	CW	L	CL
1.	0.031	0.031	0.144	0.144	0.336	0.336	1.675	1.675
2.	0.051	0.041	0.251	0.197	0.348	0.342	1.748	1.711
3.	0.035	0.039	0.167	0.187	0.347	0.344	1.719	1.714
4.	0.030	0.037	0.137	0.175	0.323	0.339	1.577	1.680
5.	0.034	0.036	0.163	0.172	0.320	0.335	1.607	1.666
6.	0.026	0.034	0.111	0.162	0.322	0.333	1.592	1.653
7.	0.027	0.033	0.122	0.156	0.311	0.330	1.531	1.636
8	0.038	0.034	0.171	0.158	0.338	0.331	1.665	1.639
9.	0.069	0.038	0.329	0.177	0.386	0.337	1.975	1.677
10.	0.038	0.038	0.167	0.176	0.334	0.337	1.612	1.670
11.	0.049	0.039	0.227	0.181	0.347	0.338	1.724	1.675
12.	0.048	0.039	0.232	0.185	0.351	0.339	1.722	1.679
13.	0.046	0.040	0.214	0.187	0.351	0.340	1.696	1.680
14.	0.047	0.040	0.214	0.189	0.355	0.341	1.736	1.684
15.	0.034	0.040	0.162	0.187	0.311	0.339	1.542	1.675
16.	0.051	0.041	0.247	0.191	0.347	0.339	1.718	1.678
17.	0.051	0.041	0.240	0.194	0.356	0.340	1.766	1.683
18.	0.030	0.041	0.134	0.191	0.326	0.339	1.605	1.678
19.	0.053	0.041	0.265	0.195	0.348	0.339	1.843	1.687
20.	0.033	0.041	0.115	0.193	0.321	0.340	1.603	1.683
21.	0.051	0.041	0.224	0.194	0.344	0.339	1.680	1.683
22.	0.040	0.041	0.172	0.193	0.350	0.340	1.636	1.681
23.	0.083	0.043	0.417	0.203	0.397	0.342	2.040	1.696
24.	0.029	0.043	0.137	0.200	0.308	0.341	1.512	1.689
25.	0.044	0.043	0.215	0.201	0.341	0.341	1.696	1.689
26.	0.028	0,042	0.119	0.197	0.317	0.340	1.476	1.681
27.	0.047	0.042	0.201	0.198	0.353	0.340	1.649	1.679
28.	0.038	0.042	0.174	0.197	0.342	0.340	1.656	1.679
29.	0.051	0.042	0.237	0.198	0.354	0.341	1.753	1.681
30.	0.072	0.043	0.351	0.203	0.375	0.342	1.878	1.688

Table C.12

Results of simulation runs

M = 3 ρ = 0.5 N = 1000

No.	W_q	CW_q	L_q	CL_q	W	CW	L	CL
1.	0.042	0.042	0.199	0.199	0.328	0.328	1.620	1.620
2.	0.048	0.045	0.224	0.212	0.350	0.339	1.751	1.686
3.	0.059	0.050	0.293	0.239	0.380	0.353	2.017	1.796
4.	0.048	0.049	0.219	0.234	0.340	0.350	1.624	1.753
5.	0.039	0.047	0.186	0.224	0.338	0.347	1.676	1.738
6.	0.042	0.046	0.197	0.220	0.345	0.347	1.730	1.736
7.	0.049	0.047	0.247	0.224	0.346	0.347	1.773	1.742
8.	0.067	0.049	0.319	0.236	0.380	0.351	1.889	1.760
9.	0.049	0.049	0.240	0.236	0.353	0.351	1.798	1.764
10.	0.038	0.048	0.173	0.230	0.336	0.350	1.644	1.752
11.	0.083	0.051	0.378	0.243	0.406	0.355	1.927	1.768
12.	0.052	0.051	0.246	0.243	0.341	0.354	1.695	1.762
13.	0.058	0.052	0.291	0.247	0.358	0.354	1.897	1.772
14.	0.040	0.051	0.183	0.243	0.333	0.353	1.679	1.766
15.	0.030	0.050	0.134	0.235	0.323	0.351	1.561	1.752
16.	0.051	0.050	0.244	0.236	0.355	0.351	1.799	1.755
17.	0.094	0.052	0.477	0.250	0.415	0.355	2.147	1.778
18.	0.043	0.052	0.205	0.248	0.352	0.354	1.738	1.776
19.	0.029	0.051	0.132	0.241	0.321	0.353	1.584	1.766
20.	0.042	0.050	0.186	0.239	0.337	0.352	1.596	1.758
21.	0.044	0.050	0.200	0.237	0.356	0.352	1.764	1.758
22.	0.038	0.049	0.181	0.234	0.340	0.352	1.714	1.756
23.	0.030	0.048	0.142	0.230	0.329	0.351	1.656	1.752
24.	0.054	0.049	0.254	0.231	0.370	0.351	1.861	1.756
25.	0.044	0.048	0.207	0.230	0.347	0.351	1.734	1.755
26.	0.075	0.050	0.373	0.236	0.378	0.352	1.953	1.763
27.	0.048	0.049	0.231	0.236	0.343	0.352	1.755	1.763
28.	0.034	0.049	0.150	0.233	0.343	0.352	1.625	1.758
29.	0.048	0.049	0.224	0.232	0.353	0.352	1.754	1.758
30.	0.068	0.050	0.321	0.235	0.368	0.352	1.882	1.762

Appendix D

Sample 't' test

The results obtained from the first ten runs are taken from Table C.6, and the standard deviation is calculated.

No.	X_i	X_i^2
1	1.007	1.014
2	1.025	1.051
3	0.907	0.823
4	1.039	1.080
5	0.991	0.982
6	0.837	0.701
7	0.784	0.615
8	1.305	1.703
9	1.019	1.038
10	1.066	1.136
	$X_i = 9.980$	$X_i^2 = 10.142$

$$\text{Mean } \bar{x} = \frac{\sum X_i}{10} = 0.998$$

$$\text{Variance} = \frac{1}{n-1}\left[\sum X_i^2 - \frac{(\sum X_i)^2}{n}\right]$$

$$= \frac{1}{9}\ (10.142 - 9.96)$$

$$= \frac{0.182}{9}$$

$$= 0.020$$

Standard deviation,

$$s = \sqrt{0.020}$$
$$= 0.140$$

t test:

A = population value of the statistic

= 1.00

α = level of significance

= 0.05

ν = degrees of freedom = n-1 = 10-1 = 9

$$t = (\bar{x} - A) / (s/\sqrt{n})$$

$$= \frac{0.998 - 1.000}{0.141/\sqrt{10}}$$

$$= 0.045$$

Theoretical $t_{.05,9} = 2.262$[1]

Calculated $t < t_{.05,9}$

The calculated value of t is not significant, That is, the cumulative average is not different from the true value of the statistic.

1. The theoretical value of 't' is obtained from Table A.3 in Reference 10.

Table D.1

Calculated values of t

M=1 ρ =0.5

Statistic	n	N = 100	200	400	600	800	1000
W_q	5	0.736	0.884	1.039	7.362*	0.719	1.413
	10	1.470	0.970	1.625	1.084	0.889	0.220
	15	0.471	0.550	0.430	1.763	0.737	0.244
	20	0.595	0.756	0.505	2.326*	0.534	0.563
	25	0.761	0.928	0.249	2.883*	0.587	0.404
	30	0.640	1.270	0.244	2.997*	1.055	1.056
L_q	5	0.785	1.186	0.952	6.468*	0.012	1.810
	10	1.531	0.916	0.853	1.473	0.674	0.592
	15	0.634	0.403	0.090	2.225*	0.729	0.210
	20	0.865	0.609	0.182	2.836*	0.171	0.016
	25	0.948	0.953	0.980	3.360*	0.712	0.332
	30	0.781	0.184	0.839	3.460*	1.338	0.096
W	5	0.620	0.751	1.036	7.893*	1.751	1.564
	10	1.096	0.910	1.908	1.150	0.201	0.158
	15	0.586	0.502	0.524	1.843	0.339	0.245
	20	0.752	0.754	0.626	2.470*	0.296	0.462
	25	0.997	0.836	0.071	2.975*	0.248	0.383
	30	0.919	1.155	0.111	2.934*	0.493	1.010
L	5	0.598	1.312	1.643	5.538*	0.899	0.268
	10	0.969	0.580	1.800	0.862	0.013	0.045
	15	0.838	0.039	0.551	1.509	0.255	0.554
	20	1.244	0.215	0.605	1.835	0.849	0.827
	25	1.330	0.466	0.135	2.175*	0.358	0.599
	30	1.219	0.577	0.158	2.168*	0.014	1.270

Theoretical values of t

df	4	9	14	19	24	29
t	2.776	2.262	2.145	2.093	2.064	2.045

*t is significant at .05 level.

The theoretical values are obtained from Table A.3 in Reference 10.

Table D.2

Calculated values of t

M=3 ρ =0.5

Statistic	n	N = 100	200	400	600	800	1000
W_q	5	0.391	0.191	1.946	1.532	2.832*	0.058
	10	0.932	1.469	1.381	0.919	2.205	0.384
	15	0.927	0.878	0.628	1.168	2.296*	0.765
	20	0.428	1.377	1.853	0.271	2.377*	0.875
	25	0.574	1.402	0.657	0.505	1.594	0.502
	30	0.234	0.211	1.274	0.042	1.406	0.909
L_q	5	0.334	0.476	2.256	2.933*	3.161*	0.691
	10	0.913	2.119	2.143	1.561	2.918*	0.494
	15	0.867	1.260	1.634	1.917	3.353*	0.106
	20	0.447	1.916	2.995*	0.911	3.486*	0.090
	25	0.554	2.073*	1.451	1.220	2.619*	0.436
	30	0.199	1.023	2.107*	0.795	2.597*	0.126
W	5	1.048	0.172	2.317	0.193	2.087	0.022
	10	1.368	0.928	0.389	0.401	1.572	0.468
	15	1.169	0.258	0.492	0.486	1.633	0.597
	20	0.662	0.261	0.314	0.425	2.026	0.853
	25	0.995	0.428	0.230	0.122	1.515	0.884
	30	0.668	0.947	0.590	0.587	1.341	1.262
L	5	0.791	0.358	1.431	1.076	2.215	0.008
	10	1.095	1.645	1.362	1.120	1.679	0.381
	15	0.952	0.724	0.185	1.255	2.190*	0.445
	20	0.519	1.325	1.110	0.099	2.268*	0.603
	25	0.750	1.106	0.115	0.047	1.927	0.653
	30	0.173	0.137	0.768	0.159	2.074*	0.677

Theoretical values of t

df	4	9	14	19	24	29
t	2.776	2.262	2.145	2.093	2.064	2.045

*t is significant at .05 level.

The theoretical values are obtained from Table A.3 in Reference 10.

Table D.3

Results of the test on confidence

M=1 ρ =0.5

IL = 1.96 σ spread of the distribution is within the limits
OL = 1.96 σ spread of the distribtuion is outside the limits.

	n	N = 100	200	400	600	800	1000
W_q	5	OL	OL	OL	OL	OL	OL
	10	OL	OL	OL	OL	OL	OL
	15	OL	OL	OL	OL	OL	OL
	20	OL	OL	OL	OL	OL	OL
	25	OL	OL	IL	OL	OL	IL
	30	OL	OL	OL	OL	OL	IL
L_q	5	OL	OL	OL	OL	IL	OL
	10	OL	OL	OL	OL	OL	OL
	15	OL	OL	OL	OL	IL	OL
	20	OL	OL	OL	OL	IL	IL
	25	OL	OL	OL	OL	IL	IL
	30	OL	OL	OL	OL	IL	IL
W	5	OL	OL	OL	OL	IL	IL
	10	OL	OL	OL	IL	IL	IL
	15	OL	OL	IL	IL	IL	IL
	20	OL	OL	IL	IL	IL	IL
	25	OL	OL	IL	IL	IL	IL
	30	OL	OL	IL	IL	IL	IL
L	5	OL	OL	OL	OL	IL	IL
	10	OL	OL	OL	OL	IL	IL
	15	OL	OL	IL	OL	IL	IL
	20	OL	OL	IL	IL	IL	IL
	25	OL	OL	IL	IL	IL	IL
	30	OL	OL	IL	IL	IL	IL

Table D.4

Results of the test on confidence

M=3 ρ =0.5

IL = 1.96 σ spread of the distribtuion is within the limits.
OL = 1.96 σ spread of the distribution is outside the limits.

	n	N = 100	200	400	600	800	1000
W_q	5	OL	OL	OL	OL	OL	OL
	10	OL	OL	OL	OL	OL	OL
	15	OL	OL	OL	OL	OL	OL
	20	OL	OL	OL	OL	OL	OL
	25	OL	OL	OL	OL	OL	OL
	30	OL	OL	OL	OL	OL	OL
L_q	5	OL	OL	OL	OL	OL	OL
	10	OL	OL	OL	OL	OL	OL
	15	OL	OL	OL	OL	OL	OL
	20	OL	OL	OL	OL	OL	OL
	25	OL	OL	OL	OL	OL	OL
	30	OL	OL	OL	OL	OL	OL
W	5	OL	IL	IL	IL	IL	IL
	10	OL	IL	IL	IL	IL	IL
	15	OL	IL	IL	IL	IL	IL
	20	OL	IL	IL	IL	IL	IL
	25	OL	IL	IL	IL	IL	IL
	30	OL	IL	IL	IL	IL	IL
L	5	OL	IL	IL	IL	IL	IL
	10	OL	OL	IL	IL	IL	IL
	15	OL	IL	IL	IL	IL	IL
	20	OL	IL	IL	IL	IL	IL
	25	OL	IL	IL	IL	IL	IL
	30	OL	IL	IL	IL	IL	IL